Paper

The Oscillatory Features of Triangular and Square Prism Oscillator Networks

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Synopsis: Oscillator networks in the form of triangular and square prisms are investigated. These kinds of networks can be represented by regular graphs of degree 3. The oscillations that appeared in the two colorable graph type of network consisting of passive oscillators were rigidly stable for parameter changes. Oscillations resembling spindle-like waves are found in two layered networks consisting of passive oscillators. Networks consisting of active oscillators keep the characteristics of each active oscillator, but characteristic oscillations appear at lower parameter values than those that appear in isolated oscillators.

Key words: Oscillator network, 3–regular graph, 1–D oscillator, two colorable graph

1. Introduction

We designed two types of oscillatory elements using one-dimensional mapping of a cubic function [1]. Two types, p– and n– type of 1–D oscillators, result for positive and negative signs of the cubic function $f(x) = Ax(1 - x^2) + x$. In the present paper, we investigate oscillatory features for networks where those 1–D oscillators are connected to each other. The previous study [1] was devoted to investigating the fundamental properties of those two types of 1–D oscillators and the control characteristics obtained by connecting those oscillators. We also studied a network of four n– and p– oscillators [2] where the interaction among 1–D oscillators was taken as the average of the inputs from other oscillators, in other words, each 1–D oscillator was totally governed by its neighboured oscillators, implying no *self quantity* in the oscillator function.

The current study seeks to find the oscillatory properties of prismic networks. The structures of prisms are isomorphic to the regular graphs of degree 3 (hereinafter referred to as 3–regular graphs, following the graph theory terminology of the textbook by R. J. Wilson [3]). Thus it becomes an interesting subject for us as to whether or not the graph structure in a network reflects on the oscillatory features of the network. Polygon prisms of an even number corners can be made such that no neighbouring oscillators are of the same type. This means

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that the polygon prisms of even numbers of corners reduce to two colorable graphs [3]. In the case where each oscillator is passively driven only by the average input level incoming from the connected oscillators, it is expected from previous study of four oscillator networks that the two colorable graph type of networks will an oscillatory nature of a total network system, such that the nonlinearity realizes pulse-like oscillations. We also examine the oscillatory features of two layered networks where one layer is a formed with one kind of oscillators, and the other layer composed of the opposite type of oscillators. Moreover, every oscillator of each network layer is connected to the corresponding oscillator of the other layer in the manner of a 3–regular graph.

The present study also investigates the difference of oscillatory features between active and passive characteristics of the oscillator properties. We say that a oscillator is active for the input-output relations of the form $O(x)=f(x, A)+\mu(I(x))$, and a oscillator is passive when the input-output relation has the form $O(x)=f(\mu(I(x, A)))$, where $\mu(I(x, A))$ indicates an operation for the input quantity $I(x)$, $O(x)$ denotes output, and $I(x)$ is the input. In the latter case, namely, passively behaving oscillators, these systems organize into synchronous oscillations of each element of a two-colorable oscillator network, that is to say a large-scale oscillator as operating as a network system.

In section 2, we discuss the general properties of prismic networks from the viewpoint of graph theory, and note that coloring graphs of two colors are possible for prismic networks of even numbers of polygon corners. Section 3 shows the oscillations of prismic networks that define a class of 3–regular graphs. Two input-output relations of active and passive oscillator properties are simulated numerically for two layered networks and two-colorable networks. Here we examine networks of triangular and square prisms only. In section 4 we discuss generality of the results obtained in section 3, and a few applications of the oscillator networks considered here.

2. Network Property of Prism Shapes

Networks of prismic form can be represented as 3–regular graphs. Then, we see that prismic forms of networks are isomorphic to the 3–regular graphs. Thus, we take some ideas from the graph theory. The arrangement of different types of oscillators in a network becomes the same as the question of whether or not the graph presented is colorable with n colors. Graph theory can supply all the different types of networks composed with n different types of network elements. We therefore investigate the general properties of a class of networks characterized by a graph property.

In the 3–regular graph type of networks, the 1–D oscillators are located at each corner point of a prism and output-input connections (sides or edges of a prism) operate both ways between pairs of neighbouring oscillators. Examples are seen in (a) and (b) of Fig. 1. Taking input-output relations between connections of neighbouring oscillators into account, the graph for these kinds of networks becomes an oriented one. In the present study, it is not so

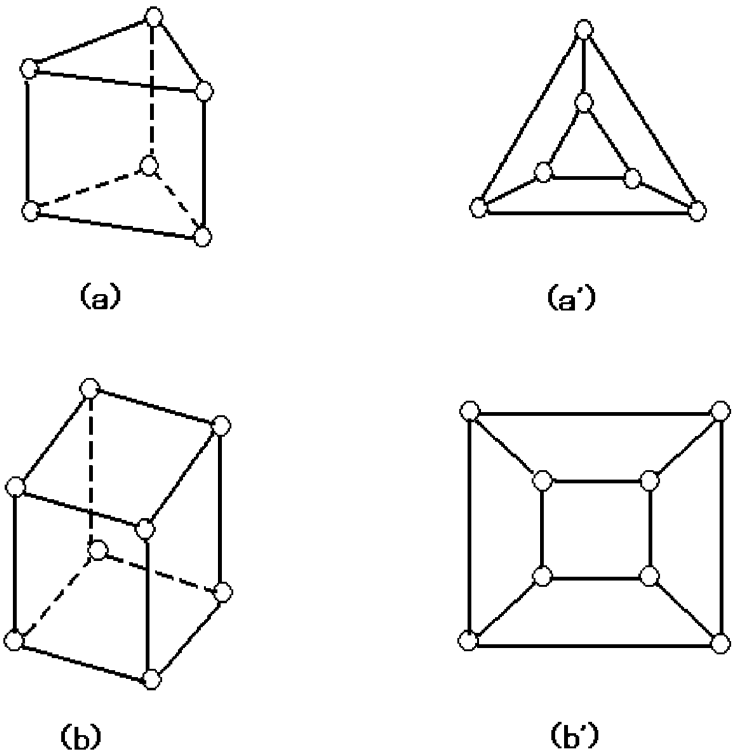

Fig. 1 Graph representation triangular and square prisms
The right side pictures (a′) and (b′) are graphs for triangular prism (a) and square prism (b).

important whether the graph is oriented or not, because input-output connections always operate both ways for every pair of neighbouring oscillators. Here we consider networks consisting of p– and n– types of oscillators so that the two-colorable property of 3–regular graphs becomes of interest for different arrangements of oscillator types. Thus, we examine how the oscillatory features change for different arrangements of the two types of oscillators. This point is well related to the problem of whether a graph is two-colorable or not.

As seen from Fig. 2, we know only that the 3–regular graph corresponding to the polygon prism of an even number of corners can be colored completely with two different colors. The polygon prism of an odd number of corners will frustrate attempts to color a 3–regular graph corresponding to the odd polygon prism by two different colors. It is therefore clearly expected that the oscillatory features are different in the oscillator networks of the polygon prisms for even and odd numbers of corners. Oscillator networks of even corner-numbered polygonal prisms can be regarded as networks of four oscillator groups which were studied in ref. [2]. Thus similar oscillatory properties are expected even though oscillatory features are governed only by the parameter A in the 1–D oscillator.

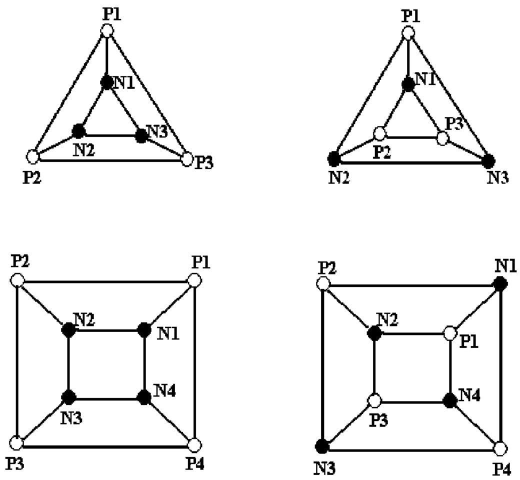

Fig. 2 Two layered networks and two colored networks
The left side is two layered networks and the right side two colorable graph type networks. The two colorable graph type of network is shortly called two colored network. The P1 to P3 or P4 are numbering for p–type oscillators and the N1 to N3 or N4 are those for n–type oscillators. Oscillations of oscillators are recognized by P1, P2, ..., and N1, N2, ...in the figures for oscillation features.

3. Oscillatory Features for the Networks in the Shapes of Triangular and Square Prism

The triangular prismic network consists of six 1–D oscillators, one at each corner. This network has the same number of the two types of oscillators, i.e., three p–type oscillators and three n–type oscillators. Similarly the square prism shape consists of four p–type oscillators and four n–type oscillators. The networks studied in this paper are those shown in Fig. 2.

To consider how the properties of 1–D oscillator networks depend on the output-input relation between connected neighbouring oscillators, two kinds of output-input relations were taken into the account in the one thought. One is additive input to each oscillator which has self-sustaining oscillations with a parameter, and another is that every oscillator governed by the inputs from connected oscillators, namely, each oscillator has no self quantities. We call former type oscillator network an active oscillator network, and latter type a passive oscillator network.

The numerical calculations in this study were carried out changing the parameter value A for each of the nonlinear 1–D oscillators. The p–type and n–type oscillators in each network make pairs like (P1, N1), (P2, N2), et al. The procedure of parameter changes is follows : Select a set of p–type and n–type oscillators to change the parameter A in the same manner, and fix the parameters of the other oscillators with small deviations. Thus the parameter changes occur at the same time for each pair of p–type and n–type oscillators. The initial values of oscillating quantities are the same for all 1–D oscillators in the network.

In the present section, we show the numerical results for thee property sets, namely, (two layered, two colored), (triangular prism, square prism) and (active, passive). Here examples of oscillatory features are shown for those 8 cases obtained by combination of the three independent properties.

3.1 Active Oscillator System

The active oscillator considered here is described by the following recurrence equation,

$$x_j(t+1) = \pm A(x_j^2(t)-1)x_j(t) \pm x_j(t) \pm \frac{1}{\sum_k \varepsilon_{jk}} \sum_k \varepsilon_{jk} x_k(t), \quad (1)$$

where $x_j(t)$ denotes the oscillating quantity of the j–th oscillator at time t, the sign $\pm$ signifies that the oscillator is p–type or n–type, namely, $+$ for p–type and $-$ for n–type, and ε_{jk} indicates a connection marking quantity using the integer 1 or 0 like a Kronecker delta, that is, the integer 1 is given when k–th oscillator connects to j–th oscillator and the integer 0 for no connecting oscillator pairs. We note that $\varepsilon_{jj}=0$ for 3–regular graph type of networks. The last term of eq. (1) is the input averaged over the outputs from oscillators connected. It is immediately apparent that, $\sum_k \varepsilon_{jk}=3$, for the networks corresponding to 3–regular graphs.

3.1.1 Two layered networks

The oscillatory features of both triangular and square prisms are similar to each other. The realized oscillatory features are similar to those of the originally designed oscillator (see ref. 1), but the parameter area becomes lower in value as the network size becomes larger. The commonly observed features of these kinds of networks are shown in Figs. 3 and Fig. 4. A special oscillation that is seen in the triangular prism network is shown in Fig. 5. Fig. 6 illustrates an example of chaotic oscillations of the square prism network. The square prism network behaves like as if P3 is similar to or the same as P1, and N3 is similar to or the same as N1 as seen from Figs. 4 and 6.

3.1.2 Two Colored Networks

In triangular prim networks that are two-colorable with respect to the oscillator types, the oscillatory features are similar to those observed in the two layer type of triangular prism networks, since two-colorable is impossible in this network type. As already stated in section 2, two-colored network property of the triangular prism is frustrated. This fact is seen in Fig. 7, and it predicts that new materials will not appear for triangular prism networks. It is, however, expected that some small differences can be seen between two layer graph and two colorable graph types. Such a difference is seen in the state differences of N1 by comparing Fig. 7 with Fig. 3. Another thing is that disorder is more common in the two-colored networks as shown in Fig. 8. The parameters are the same as those in Fig. 5.

In the two colorable graph type of square prism networks, the situation becomes differ-

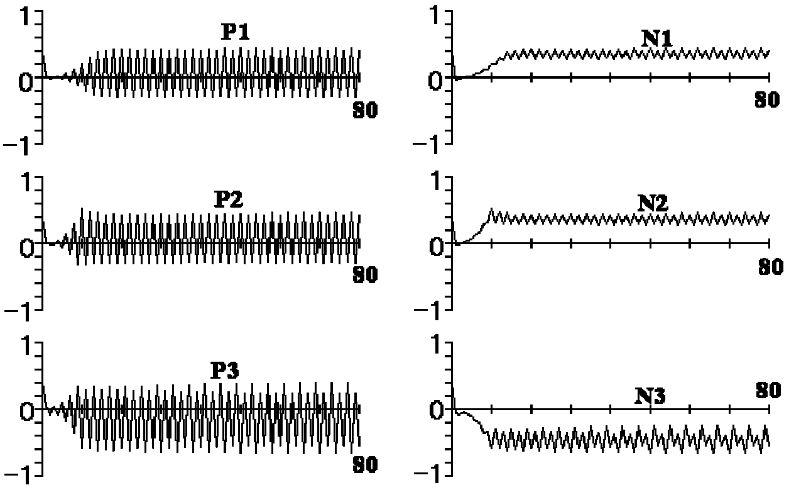

Fig. 3 Oscillation example for two layered network of active oscillators in triangular prism
The parameters of oscillators are three A_1, A_2, and A_3 to which paired p–type and n–type oscillators are corresponded, namely, A_1 for (P1, N1), A_2 for (P2, N2), and A_3 for (P3, N3). The used parameter values are $A1=2.28$, $A2=2.3$, and $A3=2.2$, and initial conditions for oscillator variables x are $x_1(0)=0.4$, $x_2(0)=0.4$, and $x_3(0)=0.4$.

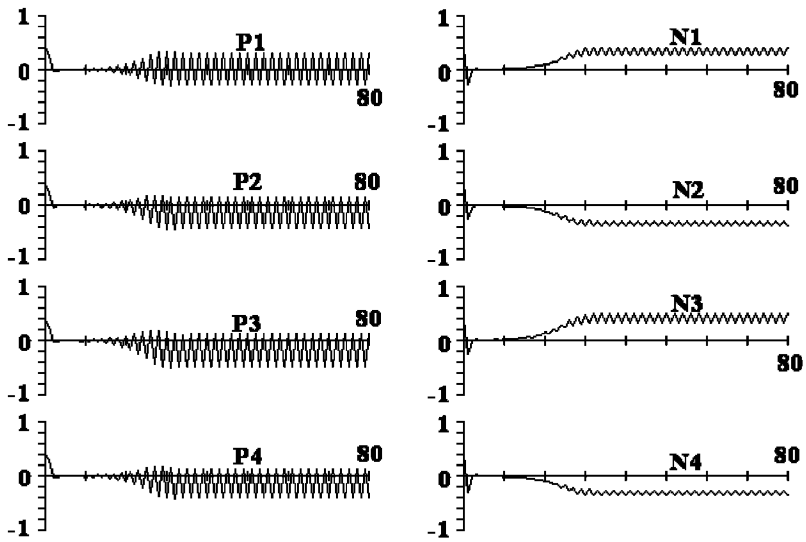

Fig. 4 Oscillation example for two layered network of active oscillators in square prism
The parameters A_1, A_2, A_3, and A_4 are apportioned to p–type and n–type oscillator pairs (P1, N1), (P2, N2), (P3, N3), and (P4, N4). The A parameters are $A_1=1.58$, $A_2=1.6$, $A_3=1.62$, and $A_4=1.5$, and initial condition for variables x are $x_1(0)=0.4$, $x_2(0)=0.4$, $x_3(0)=0.4$, and $x_4(0)=0.4$.

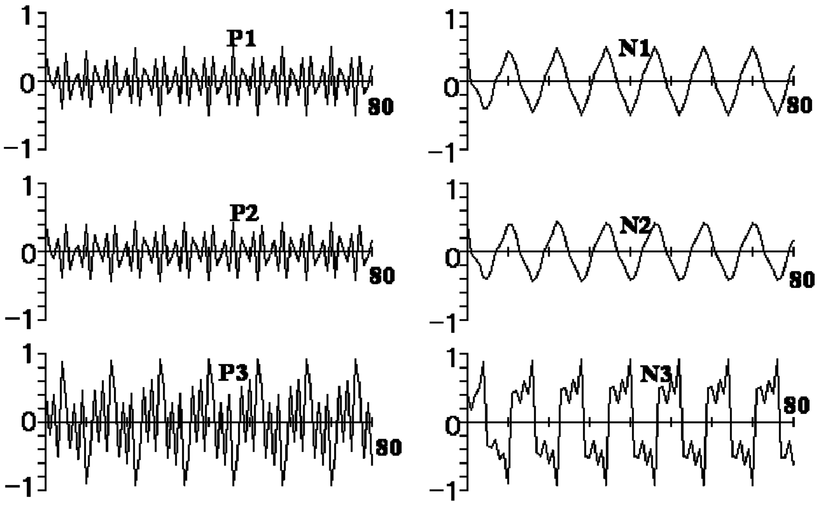

Fig. 5 Special oscillation in two layered network of active oscillators in triangular prism
The A parameters are $A_1 = 2.28$, $A_2 = 2.3$, and $A_3 = 2.9$, and initial conditions for variables x are $x_1(0) = 0.4$, $x_2(0) = 0.4$, and $x_3(0) = 0.4$.

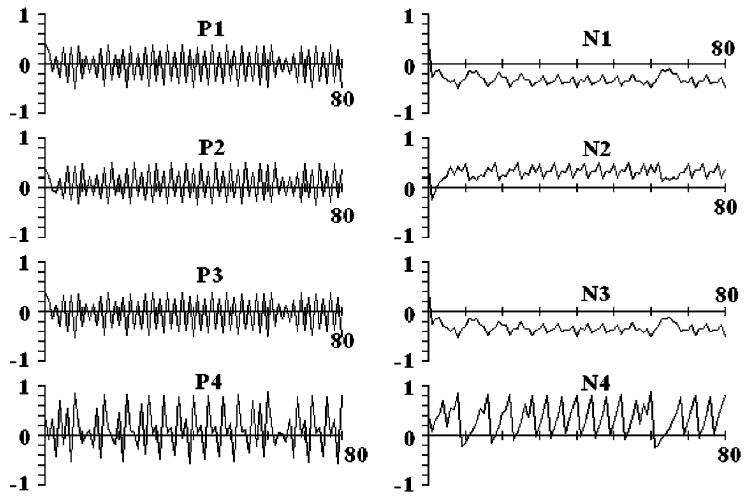

Fig. 6 Chaotic oscillation in two layered network of active oscillators in square prism
The A parameters are $A_1 = 1.58$, $A_2 = 1.6$, $A_3 = 1.62$, and $A_4 = 2.6$, and initial condition for variables x are $x_1(0) = 0.4$, $x_2(0) = 0.4$, $x_3(0) = 0.4$, and $x_4(0) = 0.4$.

ent because the square prism network can complete the coloring with two colors. The square prism network of two colors behaves well, giving ordered oscillations over a wider parameter range than other network types. An example of well ordered oscillations is depicted in Fig. 9. This kind of network shows the well similar oscillations between P1 and P3, and also be-

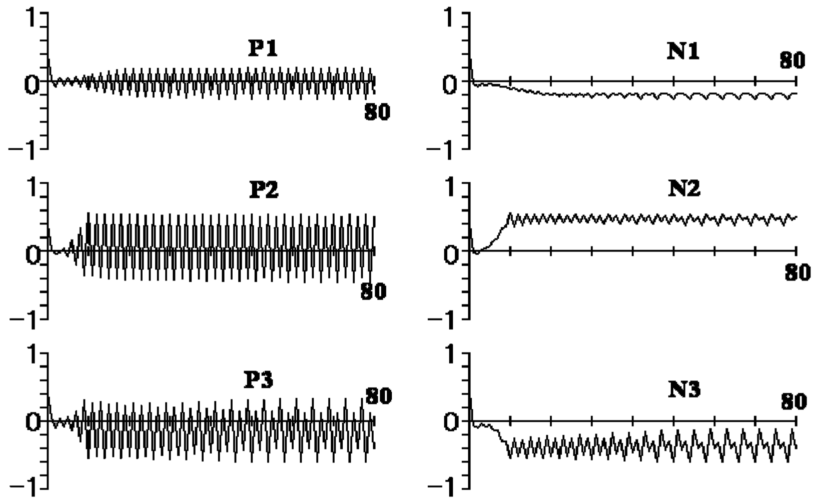

Fig. 7 Oscillation example for two colored network of active oscillators in triangular prism
The A parameters and initial conditions for variables x are the same as those in Fig. 3.

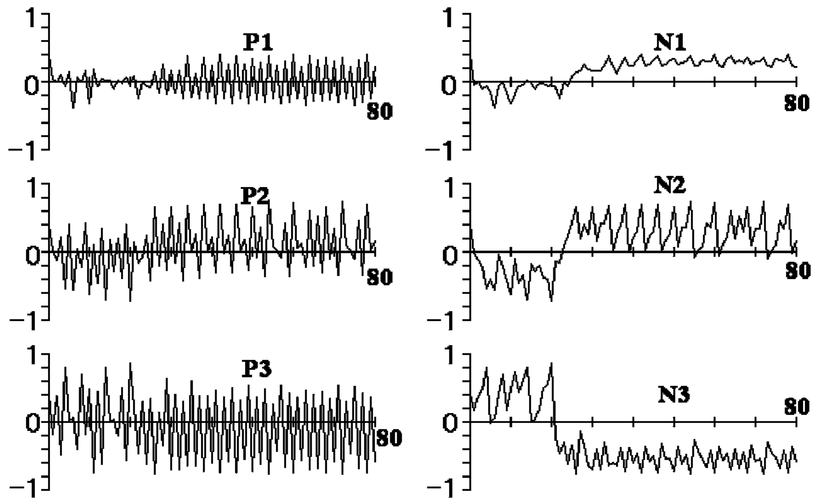

Fig. 8 Chaotic oscillation in two colored network of active oscillators in triangular prism
The A parameters and initial conditions for variables x are the same as those in Fig. 5.

tween N1 and N3. This nature is extended to chaotic oscillations are as shown in Fig. 10.

3.2 Passive Oscillator System

The passive oscillators considered here are described by the following recurrence equation,

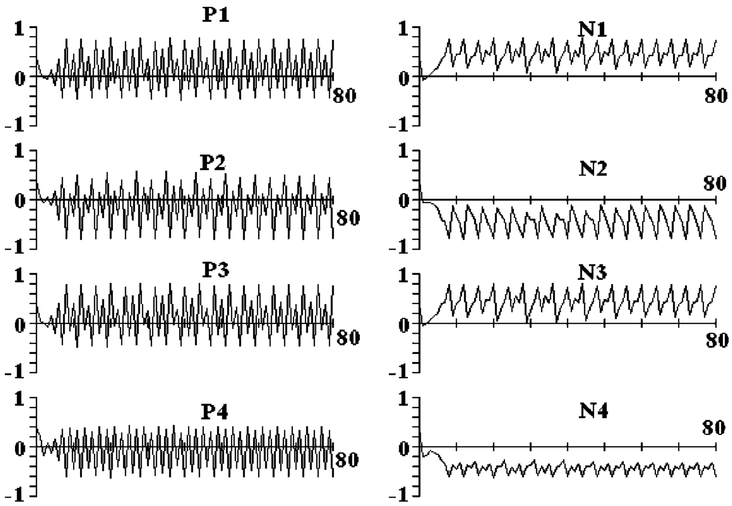

Fig. 9 Modulated oscillation in two layered network of active oscillators in square prism
The A parameters are $A_1=2.18$, $A_2=2.2$, $A_3=2.22$, and $A_4=1.8$, and initial condition for variables x are $x_1(0)=0.4$, $x_2(0)=0.4$, $x_3(0)=0.4$, and $x_4(0)=0.4$.

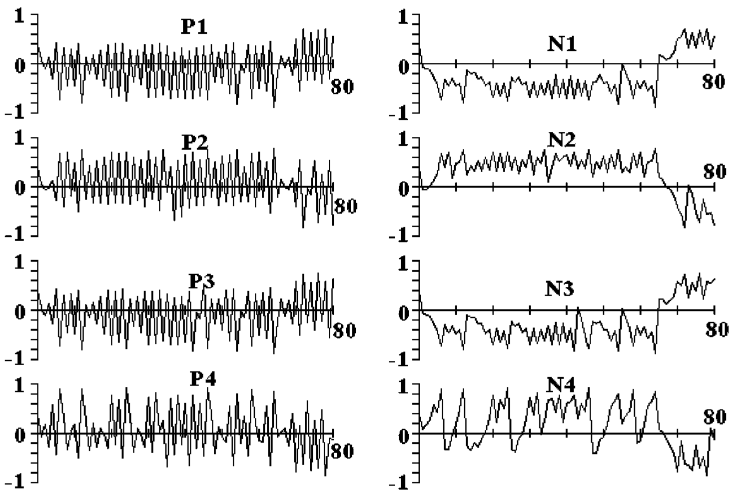

Fig. 10 Chaotic oscillation in two colored network of active oscillators in square prism
The A parameters are $A_1=2.18$, $A_2=2.2$, $A_3=2.22$, and $A_4=2.6$, and initial condition for variables x are $x_1(0)=0.4$, $x_2(0)=0.4$, $x_3(0)=0.4$, and $x_4(0)=0.4$.

$$x_j(t+1)=\pm A(I_j^2(t)-1)I_j(t)\pm I_j(t) \text{ and } I_i(t)=\frac{1}{\sum_k \varepsilon_{jk}}\sum_k \varepsilon_{jk}x_k(t), \tag{2}$$

where $x_j(t)$, sign $\pm$, and ε_{jk} are the same as those used in eq. (1). Notice again that $\varepsilon_{jj}=0$ for

3–regular graph type of networks. As stated in subsection *3.1*, 3–regular graph type of networks yields that $\sum_k \varepsilon_{jk} = 3$. As seen in eq. (2), oscillators in a network that are described by eq. (2), are operated as if the system is controlled through the other oscillators' states. In the other words, every oscillator in the network is passively governed by those oscillators that are providing outputs.

These passive oscillator networks show rather gentle oscillatory behaviours compared to those obtained for active oscillator networks. Nevertheless interesting oscillations are found in the passive oscillator networks. We can see chaotic oscillations in this kind of network, but they are not so strongly chaotic as random walk processes [4]. One interesting oscillation type which can be seen in two layered networks is spindling waves. We present below the results of oscillations for passive oscillator networks by dividing these into those for two layered networks and those for two colored networks.

3.2.1 Two Layered Networks

The oscillations in triangular prism networks are often of regular form as seen in Fig. 11, and are similar to each other for the same type oscillators. These oscillations are similar to those obtained for the isolated nonlinear 1–D oscillators used here. The oscillations still keep regularity on the chaotic region for parameter value of A as shown in Fig. 12. All the oscillators in each layer realize the same oscillation feature since the deviation of parameter values for all the oscillators is small. This implies that a synchronous nature appears in chaotic oscillations with small deviation of parameter values since the oscillation difference is seen in n–type oscillator layer for regular oscillations.

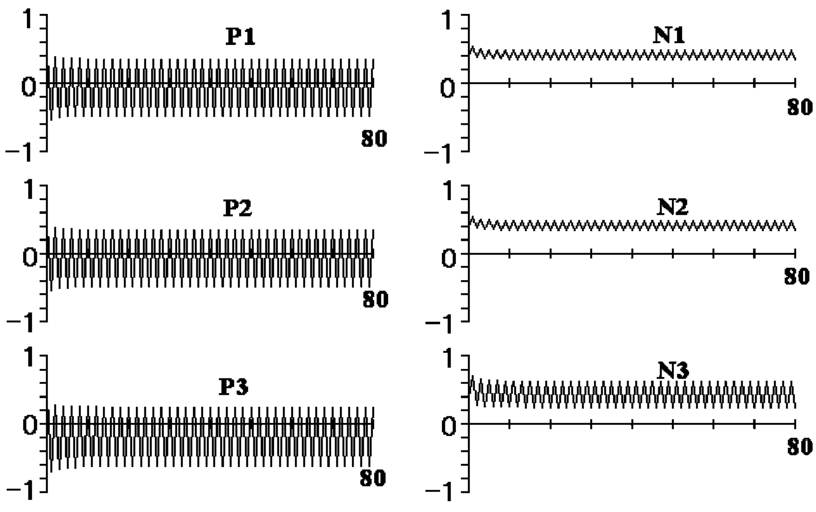

Fig. 11 Oscillation example for two layered network of passive oscillators in triangular prism
The A parameters are $A_1 = 2.78$, $A_2 = 2.8$, and $A_3 = 2.7$, and initial condition for variables x are $x_1(0) = 0.4$, $x_2(0) = 0.4$, and $x_3(0) = 0.4$.

The fundamental behaviour of the oscillations of the square prism networks is quite similar to that in triangular prism network. The oscillations that are unique to square prism networks are illustrated here, as they are interest to us. In particular, spindle-like waves were found in the two layered network of square prism shape. An example is shown in Fig. 13. These waves are seen from initial stage. The spindle-like wave means that a spindling wave is modulated on a sinusoidal curve. The period of the sinusoidal curve becomes shorter as the parameter A becomes higher. This fact can be understood from Fig. 14 by comparing the figure to the Fig. 13. The alternatively exchanging oscillation modes between p–type and n–type oscillator layers can be seen in the chaotic parameter region. This fact is seen a set figures from Fig. 15 to Fig. 17.

3.2.2 Two Colored Networks

The oscillations in two colorable graph type networks are quite rigid over a wide range of the parameter changes. Very similar oscillations to those seen in Fig. 18 for the triangular prism network and Fig. 19 for the square prism network are almost always observed. Small changes in shape and amplitude are observed for parameter changes, but the oscillations obtained are quite similar. This network type, being organization of passive oscillators, behaves like an oscillator as a whole. Thus we guess that the two colorable nature of the network is not so essential to the oscillation features produced in passive oscillator networks. Instead, the effect of two-colorable nature reflects that the oscillation properties of P1 and P3 oscilla-

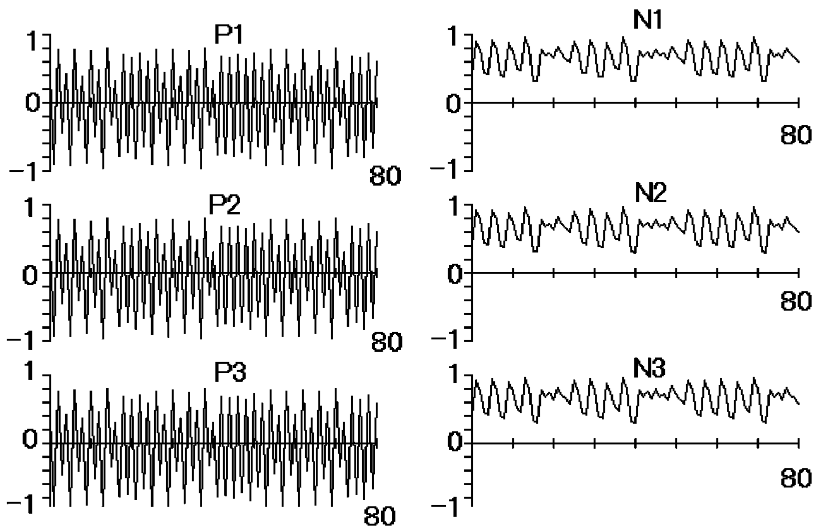

Fig. 12 Highly Modulated oscillations for two layered network of passive oscillators in triangular prism
The A parameters are $A_1 = 3.88$, $A_2 = 3.9$, and $A_3 = 3.92$, and initial condition for variables x are $x_1(0) = 0.4$, $x_2(0) = 0.4$, and $x_3(0) = 0.4$. All the values of A parameters are those in chaotic oscillation region in an isolated 1–D oscillator.

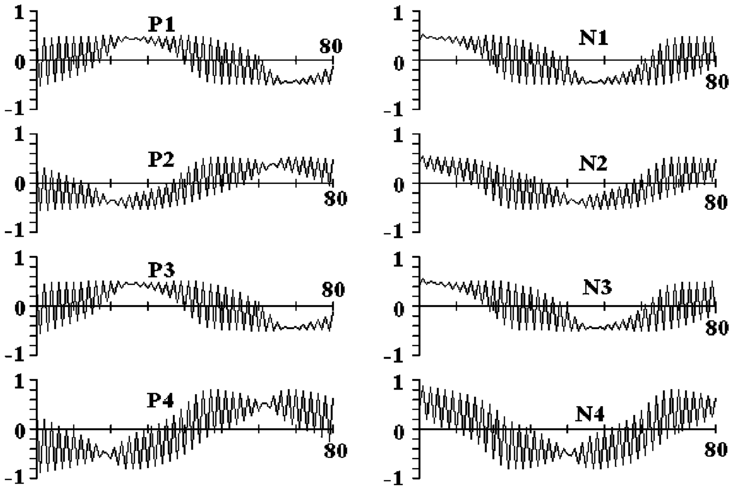

Fig. 13 Sinusoidal Spindling-waves
Spindle-like waves are observed in two layered network of passive oscillators in square prism shape. The observed spindle-like waves are sinusoidal modulations of spindle waves. The A parameters are $A_1 = 2.78$, $A_2 = 2.8$, $A_3 = 2.82$, and $A_4 = 3.8$, and initial condition for variables x are $x_1(0) = 0.4$, $x_2(0) = 0.4$, $x_3(0) = 0.4$, and $x_4(0) = 0.4$. The length of spindle wave becomes shorter for the A_4 parameter becomes higher. These spindle-like waves are stable and continuing from the initial stage.

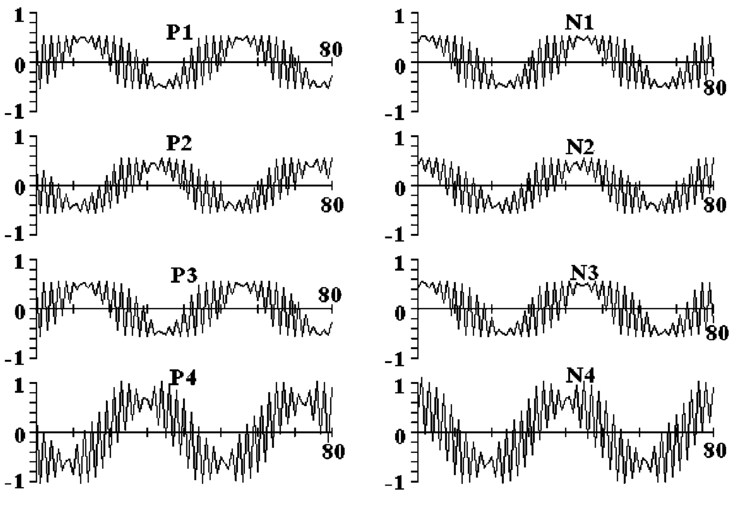

Fig. 14 Parameter dependence of sinusoidal period of the sinusoidal spindling-wave
Only the A_4 parameter is increased to a high value, $A_4 = 4.5$. All the other parameters are the same as those in Fig. 13.

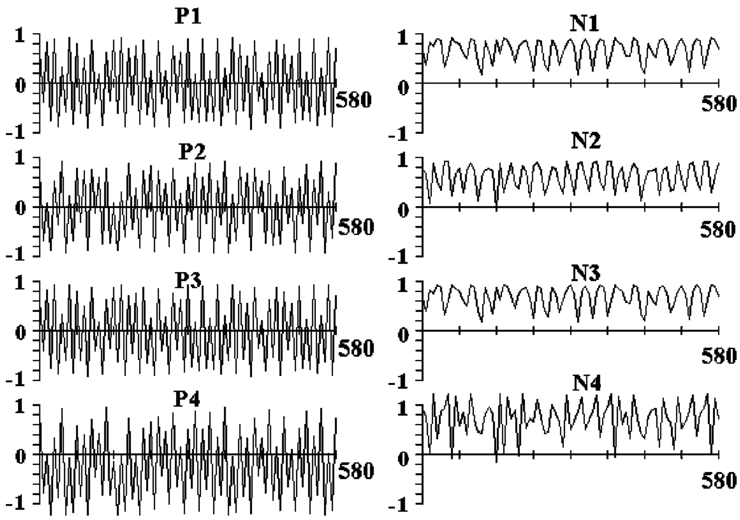

Fig. 15 Alternate oscillation modes between p-type and n-type oscillator layers (part I)
The A parameters are $A_1=3.78$, $A_2=3.8$, $A_3=3.82$, and $A_4=4.6$, and initial condition for variables x are $x_1(0)=0.4$, $x_2(0)=0.4$, $x_3(0)=0.4$, and $x_4(0)=0.4$. Usual oscillation modes can be seen in this time range (iteration time from 500 to 580).

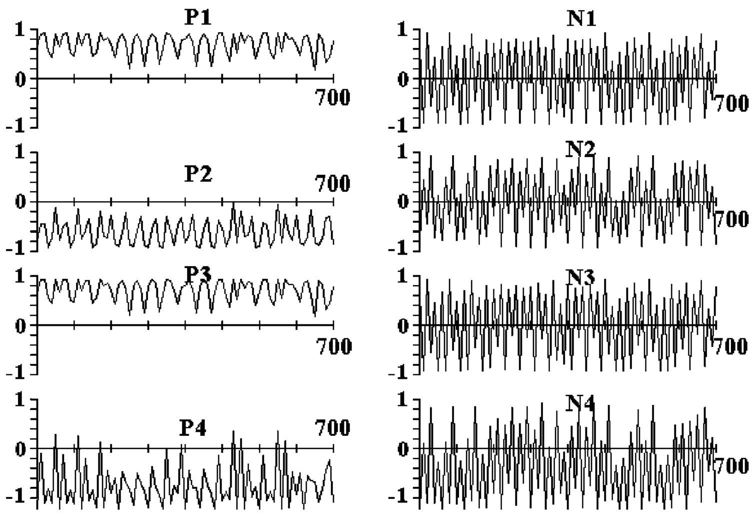

Fig. 16 Alternate oscillation modes between p-type and n-type oscillator layers (part II)
Oscillation modes are swapped in the iteration time from 620 to 700. The figures are later iteration time of oscillations for those in Fig. 15.

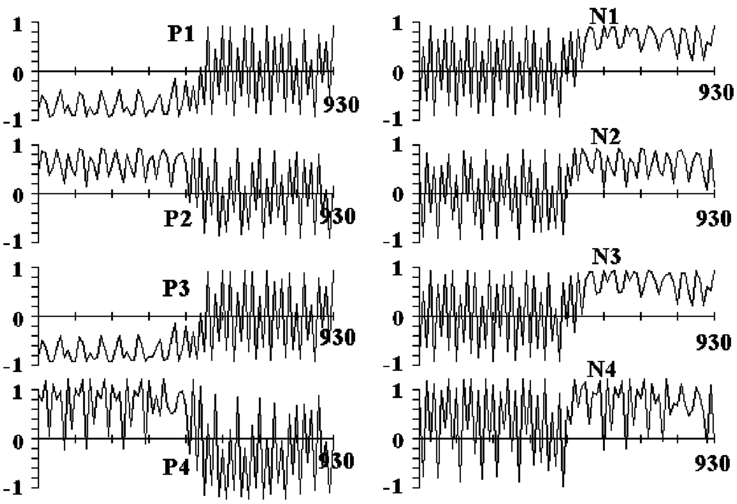

Fig. 17 Alternate oscillation modes between p-type and n-type oscillator layers (part III)
An example of alternation area of oscillation modes swapping is shown. All the parameters are the same as those in Fig. 15. The example shows that the oscillation modes return to usual modes.

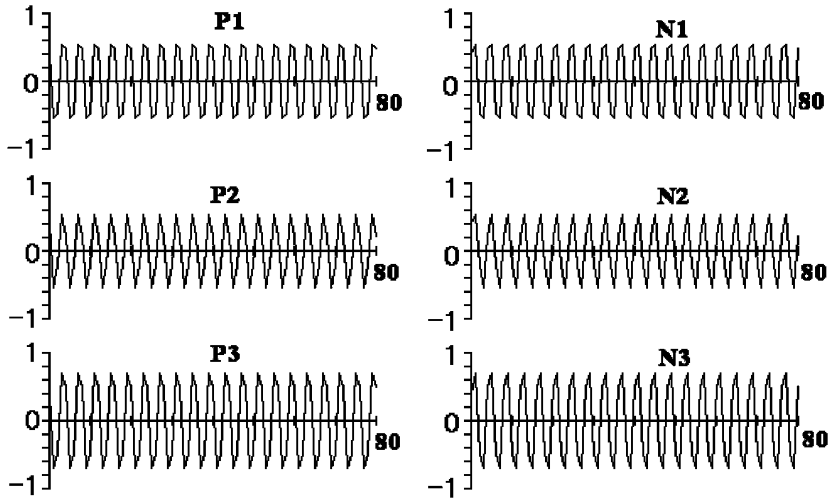

Fig. 18 Typical oscillation in two colored network of passive oscillators in triangular prism
The A parameters are $A_1 = 2.78$, $A_2 = 2.8$, and $A_3 = 2.7$, and initial condition for variables x are $x_1(0) = 0.4$, $x_2(0) = 0.4$, and $x_3(0) = 0.4$. The A parameters and initial condition for variables x are the same as those in Fig. 11.

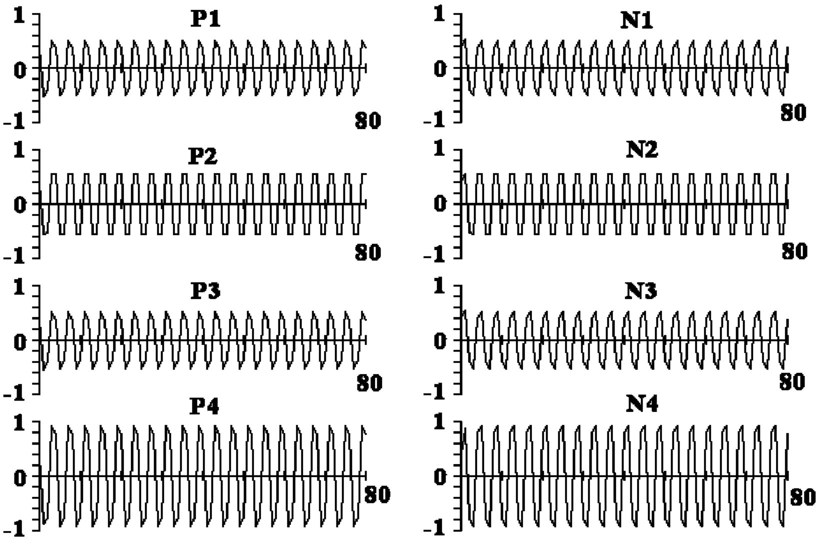

Fig. 19 Typical oscillation in two colored network of passive oscillators in square prism
The A parameters are $A_1=2.78$, $A_2=2.8$, $A_3=2.82$, and $A_4=3.8$, and initial condition for variables x are $x_1(0)=0.4$, $x_2(0)=0.4$, $x_3(0)=0.4$, and $x_4(0)=0.4$. The A parameters and initial condition for variables x are the same as those in Fig. 11 where spindle waves can be observed in two layered networks of passive oscillators.

tors and those of N1and N2 oscillators are the same or very similar to each other in the square prism network.

4. Discussions

The oscillator networks studied in the present paper are expressed as the regular graph of order three. Thus we can call the networks considered here 3–regular graph type networks. The image for of the real structure corresponding to the 3–regular graph type network is a polygon prism, namely, having oscillators are located at corners and sides (or edges) implying connection lines. The polygon prism structure supplies the application of 3–regular graph type networks for real materials or organisms.

An application is neuronally controlled behaviour of Jellyfish [5, 6]. The behaviour of Jellyfish is simple, that is, drifting or swimming in the sea. The neuronal structure of cubomedusae and hydromedusae might be reduced to a four sided polygon prism network. It is so far unknown why all these medusae have 4–fold symmetry to their major neural network elements. Usually neurons can be classified into two groups, i.e., excitatory neurons and inhibitory neurons so that the nonlinear 1–D oscillators used in this study are applicable to a modeling study of a real organism. In the oscillator model for a Jellyfish, changes of oscillation modes imply the swimming motion or behaviour switching. The spindle-like waves that appeared in two layered networks consisting of passive oscillators (Fig. 13 and 14) can perhaps be assigned to swimming modes of Jellyfish. The two layered network is appropriate to the real neuronal network of Jellyfish. We speculate that if any side of Jellyfish pushed, the parameter of pushed side oscillators becomes high and then oscillation mode goes to spindle-like waves. Our simulations assert that the push to any side causes spindle-like wave oscillation in all network elements.

Another application is the oscillations of atoms in a crystal lattice. Simple crystals have unit cells that are square prisms or cubes. That is the atoms of a simple crystal are located at the cross points of a lattice so that the unit cell is a square prism. Usually the interaction between neighbouring atoms is an additive term to the Hamiltonian of an isolated atom. Thus the network consisting of active oscillators is appropriate to making a model for such kinds of systems having output-input relations that are of the additive type. The thermal nature of crystal can be explained by n oscillator systems so that local thermal fluctuations can be modeled with such kind of oscillator networks.

Actually larger size systems are existed in real world. The general features of oscillations in two layered networks and two colored networks will be applicable in much larger networks than those examined in the present paper. It is general that oscillations of networks are similar to those observed at isolated active oscillators in the two layered networks consisting of active oscillators, except for the parameter values are lower than isolated ones. The feature found in two colored networks consisting of passive oscillators is general, that is, the oscillation of each oscillator is quite similar to each other for wide parameter range, and thus this type of networks can be regarded as an giant oscillator acting as a whole, as stated above. The two-colorable property of networks is not so characteristic. This property may produce pairing of oscillators for oscillation modes, but this point is not clear because we examined only square prism networks.

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