Paper

Discrete Algebra on Cellular Automata and Binary Textures

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Synopsis: We discuss algebras suitable for investigating both cellular automata (CA) and binary textures (Btext). A recurrence form of discrete functions of an increasing number of variables is presented. A more appropriate form, describing the rules of CA, is obtained by using the recurrence formula on the discrete functions. Concerning Btext, discrete functions on three integer sets such as $\{-1, 0, 1\}$ have more practical meaning in visually relevant textures. The discrete functions are described by using ordinal arithmetic addition and multiplication. Consideration of discrete functions leads to an algebra on sets of finite number of integers. This algebra is called 'Discrete' Algebra.

1. Introduction

We have been studying soliton-like behaviour of some cellular automata [1] and discrimination of binary textures [2, 3]. In the studies, we use mappings on sets consisting of a few integers such as $\{0, 1\}$, $\{-1, 1\}$, or $\{-1, 0, 1\}$, which we refer to as discrete mappings. These discrete mappings can be described by functions based on an algebra for a discrete set. Here a discrete set means a set consisting of a finite number of integers. All functions on a discrete set are finite so that the whole function space for a discrete set can be seen at an instant. We call the functions for discrete sets discrete functions. Discrete functions on a discrete set are related to each other much as the Boolean functions.

In the present paper we describe discrete functions using ordinal sums and products rather than Boolean expressions. The Boolean functions can also be expressible by using ordinal operations of addition and multiplication. The expression of discrete functions by use of real functions leads to a set of real functions which implements the discrete mapping completely for special points. These special points form a group, and establish a discrete algebra.

In section 2, we define the discrete algebra and discuss the nature of it. We also discuss the discrete algebra and its relationship to the Boolean algebra. In section 3, we consider practical discrete functions for sets consisting of two or three elements. The practical expressions become different, depending on what kind of integer assignment we utilize. We also

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present a recurrence formula of discrete functions with respect to increases neighbour numbers. The generalization of two and three element cases gives a theorem on discrete functions. In section 4, we discuss the application of discrete algebra to cellular automata (CA) and binary textures (Btext).

2. Discrete Algebra

As mentioned in introduction, we describe algebras on discrete sets. The discrete algebra was derived from a Boolean algebra when we studied rules on cellular automata and the nature of binary textures and human discrimination of different types of textures. In cellular automata, usually the set of integers $\{0, 1\}$ are used to represent each cell state. We however extended the state values to three integers, i.e., a set $\{-1, 0, 1\}$, when we studied solitons representing a discretized analogue of partial differential equations. The set of integers $\{-1, 1\}$ are used to make a binary texture where each pixel of the texture can be thought of as being either black or white. The set $\{-1, 1\}$ are easily converted to the set $\{0, 1\}$ by the relation $S=(1+s)/2$ or $S=(1-s)/2$, where S denotes one of $\{0, 1\}$, and s that of $\{-1, 1\}$. Then an algebra on $\{-1, 1\}$ has the same properties as the Boolean algebra on $\{0, 1\}$. But the set $\{-1, 1\}$ don't contain zero explicitly so that we can easily see that some of postulates for the set $\{0, 1\}$ aren't satisfied for the set $\{-1, 1\}$. This point lead us to consider discrete algebras.

To discuss the discrete algebras, we first mention Boolean algebra. A Boolean algebra is defined as follow [4, 5]:

Definition 2–1 (*Boolean algebra*), A class of set B together with two operations $(+)$ and $(\cdot)$ is a Boolean algebra if and only if the following postulates hold;

P_0. The elements of B are closed with respect the operations $(+)$ and$(\cdot)$, respectively.

P_1. The operations $(+)$ and $(\cdot)$ are commutative. ($X+Y=Y+X$ and $X\cdot Y=Y\cdot X$).

P_2. There exist in B distinct elements 0 and 1 relative to the operations $(+)$ and $(\cdot)$, respectively. ($X+0=X$ and $X\cdot 1=X$)

P_3. Each operation is distributive over the other.
($X+Y\cdot Z=(X+Y)\cdot(X+Z)$ and $X\cdot(Y+Z)=X\cdot Y+X\cdot Z$).

P_4. For every element X in B, there exists an element $\sim X$ in B such that
$X+\sim X=1$ and $X\cdot\sim X=0$.

Notice that X, Y, Z denote elements in B, and the set $\{0, 1\}$ make such a kind of set B.

As seen from the definition of Boolean algebra, the set $\{-1, 1\}$ lacks the distinct element 0. As described above, however, a one-to-one relationship between the sets $\{-1, 1\}$ and $\{0, 1\}$ exists. Then all mappings on the set $\{-1, 1\}$ is identical to those on the set $\{0, 1\}$. We can therefore see the Boolean algebra like nature with respect to the set $\{-1, 1\}$ even though the set $\{-1, 1\}$ do not satisfy the definition of Boolean algebra directly. At this point we consider another type of discrete algebra for the set $\{-1, 0, 1\}$. The set $\{-1, 0, 1\}$ have distinct elements 0 and 1. The element -1 cannot satisfy the postulate P_4. Thus we need to modify the definition of 'NOT' in Boolean algebra. This is another reason why we consider discrete algebra.

To obtain a well-defined discrete algebra, we can first think about Boolean functions. The operations of Boolean algebra are equivalent to the mappings on a set B, in fact, they are two Boolean functions. The operation $(\cdot)$ is the same as the multiplication of the ordinal arithmetic algebra, but another operation $(+)$ is different from addition of ordinal arithmetic. The operation $(+)$ yields $1+1=1$ so that we regard the operation $(+)$ as a nonlinear function. A practical expression for the operation $(+)$ of the Boolean algebra is $X+Y-XY$ for the set $\{0, 1\}$. This expression has a nonlinear term XY explicitly. We therefore use a few members of whole functions, instead of the operations such as $(+)$ and $(\cdot)$, in the discrete algebra. Hence, we define the discrete algebra as follows:

Definition 2–2 (*Discrete algebra*)

Let $\{D\}$ be discrete sets and let X and Y be elements of a set D. Operations for elements of a set D which are k self-mappings $\{f_0, f_1, \ldots, f_{k-1}\}$ on the set D are introduced and a class of set D with introduced operations is a discrete algebra if and only if a set of L postulates $\{P_0, P_1, \ldots, P_{L-1}\}$ hold. The postulates $\{P_0, P_1, \ldots, P_{L-1}\}$ include the following ones:

P_0. The elements of D are closed with respect to each operation given by the self-mappings $\{f_0, f_1, \ldots, f_{k-1}\}$. The number of operations is less than the number of distinct elements in D.

P_1. The self-mappings $\{f_0, f_1, \ldots, f_{k-1}\}$ are commutative.
$(f_j(X, Y)=f_j(Y, X), j=0, 1, \ldots, k-1)$

P_2. There exist in D distinct elements $\{X_0{}^*, X_1{}^*, \ldots, X_k{}^*\}$ relative to the self-mappings $\{f_0, f_1, \ldots, f_{k-1}\}$, respectively. $(f_j(X, X_j{}^*)=X, j=0, 1, \ldots, k-1)$

P_3. There exist distributive pair of operations.

P_4. For every element X in D, there exists an element $\sim X$ in D which satisfies a special condition such as 'NOT' element in a Boolean algebra.

The postulates from P_0 to P_4 are similar to those of Boolean algebra, except for the postulate P_3. For the set $\{-1, 0, 1\}$, the operations defined in the Boolean algebra are not equistrength to each other, i.e., only $X\cdot(Y+Z)=X\cdot Y+X\cdot Z$ is conserved. But if we use another type of operation, the postulate might be retained. When the functions $f_A(X, Y)$ and $f_M(X, Y)$ are used instead of the operations $(+)$ and $(\cdot)$, the distributive law in the Boolean algebra says that $f_M(X, f_A(Y, Z))=f_A(f_M(X, Y), f_M(X, Z))$ or $f_A(X, f_M(Y, Z))=f_M(f_A(X, Y), f_A(X, Z))$. The description using mappings on a set is more general and abstract, and has application to binary and ternary texture patterns.

3. Discrete Functions

For a discrete set, we can see all functions on the discrete set. Typically, these functions are imagined as mappings to combine elements to another element in a set. In the present paper, we express these functions by use of ordinal arithmetic addition and multiplication. Most of the discrete functions are nonlinear functions. Within a discrete algebra, two-variable functions are required to consider the operations on a set. Since our thoughts for discrete algebra came from investigations of cellular automata and binary textures in texture discriminations, we have to consider multi-variable discrete functions. It is very convenient if general expressions of multi-variable discrete functions can be given. For this purpose, we derive a recurrence form of multi-variable discrete functions, and give a general expression of multi-variables and any number of distinct elements given by integers. General expressions can give the complete set of rules of cellular automata by a single form, just like an algebra. Actually, we discuss the discrete functions on the set of two and three integers. Utilizing the constructions of these, we can speculate on a theorem that specifies any case of discrete functions.

3.1 On Two Elements Set

The set $\{0, 1\}$ gives ordinal Boolean functions. Here we express Boolean functions using the ordinal arithmetic operations of addition and multiplication. The one-variable case is presented in Table 1. For one-variable discrete functions, two fundamental functions ex-

Table 1 One-variable functions on $\{0, 1\}$

X		0	1	A practical form
$g(X)$	g_0	0	0	$g_0(X)=0$
	g_1	0	1	$g_1(X)=X$
	g_2	1	0	$g_2(X)=1-X$
	g_3	1	1	$g_3(X)=1$

ist, and combinations of them give other discrete functions. These functions are $g_1(X)=X$ and $g_2(X)=1-X$ because we then have $g_0(X)=g_1(X)g_2(X)$ and $g_3(X)=g_1(X)+g_2(X)$. Note that the relations are satisfied for the values 0 and 1 only. The values 0 and 1 are the solution of equations $X(1-X)=0$ and $X+(1-X)=1$, although the latter is a tautology. In the set $\{-1, 1\}$, the two fundamental functions are $g_+(X)=X$ and $g_-(X)=-X$. Two other functions are given by the operation of these fundamental functions such that $g_1(X)=g_+(X)g_+(X)=g_-(X)g_-(X)$ and $g_{-1}(X)=g_-(X)g_+(X)=g_+(X)g_-(X)$. The one-variable discrete functions on the set $\{-1, 1\}$ are shown in Table 3. Both cases have two fundamental functions and show that other functions can be obtained using the two fundamental functions. This implies that a group of discrete functions has a closed form given by two fundamental functions.

Now we proceed to two-variable discrete functions. The whole set of discrete functions of two variables are obtainable by using a recurrence equation, which is,

$$f_{ij}(X, Y)=(1-X)g_i(Y)+Xg_j(Y), \quad i, j=1, 2, \quad \text{for the set}\{0, 1\}, \tag{3.1.1}$$

$$f_{ab}(X, Y)=1/2(1-X)g_a(Y)+1/2(1+X)g_b(Y),$$
$$a, b=+, -, \text{ for the set}\{-1, 1\}. \tag{3.1.2}$$

The resulting values of a two-variable function correspond to the specified integer representation of subscripts in the same order. Note that a and b also mean subscripts taken the characters + and − insetad of integers 1 and 2 in (3.1.1), e.g., $a=+$ specifies $g_+(Y)$. The results of recurrence equations (3.1.1) and (3.1.2) are tabulated in Table 2 and 4, respectively. As seen from Eq. (3.1.1) the factor $1-X$ takes the value 1 at $X=0$ and takes the value 0 at

Table 2 Two-variable functions on {0, 1} by the recurrence relation $f_{ij}(X, Y)=(1-X)g_i(Y)+Xg_j(Y)$.

i	j	i, j by binary	$f_{ij}(X, Y)=(1-X)g_i(Y)+Xg_j(Y)$.
0	0	0 0 0 0	0
0	1	0 0 0 1	XY
0	2	0 0 1 0	$X(1-Y)=X-XY$
0	3	0 0 1 1	X
1	0	0 1 0 0	$(1-X)Y=Y-XY$
1	1	0 1 0 1	$(1-X)Y+XY=Y$
1	2	0 1 1 0	$(1-X)Y+X(1-Y)=X+Y-2XY$
1	3	0 1 1 1	$(1-X)Y+X=X+Y-XY$
2	0	1 0 0 0	$(1-X)(1-Y)=1-(X+Y)+XY$
2	1	1 0 0 1	$(1-X)(1-Y)+XY=1-(X+Y)+2XY$
2	2	1 0 1 0	$(1-X)(1-Y)+X(1-Y)=1-Y$
2	3	1 0 1 1	$(1-X)(1-Y)+X=1-Y+XY$
3	0	1 1 0 0	$1-X$
3	1	1 1 0 1	$(1-X)+XY=1-X+XY$
3	2	1 1 1 0	$(1-X)+X(1-Y)=1-XY$
3	3	1 1 1 1	$(1-X)+X=1$

Table 3 One-variable functions on {−1, 1}

X		−1	1	A practical form
$g(X)$	g_{-1}	−1	−1	$g_{-1}(X)=-1$
	g_+	−1	1	$g_+(X)=X$
	g_-	1	−1	$g_-(X)=-X$
	g_1	1	1	$g_1(X)=1$

Table 4 Two-variable functions on $\{-1, 1\}$ by the recurrence relation $f_{ab}(X, Y)=(1-X)g_a(Y)/2+(1+X)g_b(Y)/2$.

a	b	a, b by $\{-1, 1\}$				$f_{ab}(X, Y)=(1-X)g_a(Y)/2+(1+X)g_b(Y)/2$
-1	-1	-1	-1	-1	-1	-1
-1	+	-1	-1	-1	1	$-(1-X-Y-XY)/2$
-1	−	-1	-1	1	-1	$-(1-X+Y+XY)/2$
-1	1	-1	-1	1	1	X
+	-1	-1	1	-1	-1	$-(1+X-Y+XY)/2$
+	+	-1	1	-1	1	Y
+	−	-1	1	1	-1	$-XY$
+	1	-1	1	1	1	$(1+X+Y-XY)/2$
−	-1	1	-1	-1	-1	$-(1+X+Y-XY)/2$
−	+	1	-1	-1	1	XY
−	−	1	-1	1	-1	$-Y$
−	1	1	-1	1	1	$(1+X-Y+XY)/2$
1	-1	1	1	-1	-1	$-X$
1	+	1	1	-1	1	$(1-X+Y+XY)/2$
1	−	1	1	1	-1	$(1-X-Y-XY)/2$
1	1	1	1	1	1	1

$X=1$. On the other hand, the factor X takes the value 1 at $X=1$ and takes the value 0 at $X=0$. The same situation is seen in Eq. (3.1.2), that is the factor $(1-X)/2$ takes the value 1 at $X=-1$ and takes the value 0 at $X=1$, and on the other hand, the factor $(1+X)/2$ takes the value 1 at $X=1$ and the value 0 at $X=-1$. These factors thus behave like a Dirac delta function. More practically, these factors are equivalent to Kronecker deltas, although they have a form of continuous functions. These functions, however, have meaning only at specific points. These points define a group. Some of organized groups have the sense of group in mathematics.

For three variables, X, Y, and Z, we proceed in a similar approach to that indicated by Eqs (3.1.1) and (3.1.2). We introduce the following lemma for the case of three variables,

Lemma 3–1 (*Recurrence form of three-variable functions on two elements set*)

Let X, Y, and Z be elements on the set of two integers. There exist $2^4(=16)$ two variable discrete functions denoted by $\{f_0, f_1, \ldots, f_{15}\}$ and two orthogonal functions of $\{I_0, I_1\}$ which are point functions of one variable, like the Kronecker delta. Then all the discrete functions ($256=2^w$, $w=2^3$)of three variables are completely described by the following recurrence equation:

$$F_{ij}(X, Y, Z)=I_0(X)f_i(Y, Z)+I_1(X)f_j(Y, Z). \tag{3.1.3}$$

Two-variable discrete functions are given in Table 2 and 4. There we admit the existence of

16 two-variable functions. The recurrence form has the same sense as in previous considerations above: two orthogonal functions are obtainable by the combination of one-variable discrete functions. The actual form of these two orthogonal functions is already known above as the factors of recurrence equations (3.1.1) and (3.1.2).

3.2 On Three Elements Set

First, we consider the discrete functions on the set $\{-1, 0, 1\}$. There are $27(=3^3)$ one-variable functions on $\{-1, 0, 1\}$ as seen in Table 5. As known from the previous subsection, the fundamental functions play an important role in organizing multi-variable discrete functions. The fundamental functions on the set of three elements are three orthogonal functions. For the set $\{-1, 0, 1\}$, these are g_1, g_3, and g_9. The practical expression of g_1, g_3, and g_9 are shown in Table 5, i.e., $g_1(X)=(X^2+X)/2$, $g_3(X)=1-X^2$, and $g_9(X)=(X^2-X)/2$.

Table 5 One–varible functions on $\{-1, 0, 1\}$

X		-1	0	1	A practical form
$g(X)$	g_{-13}	-1	-1	-1	$g_{-13}(X)=-1$
	g_{-12}	-1	-1	0	$g_{-12}(X)=-1+(X+X^2)/2$
	g_{-11}	-1	-1	1	$g_{-11}(X)=X^2+X-1$
	g_{-10}	-1	0	-1	$g_{-10}(X)=-X^2$
	g_{-9}	-1	0	0	$g_{-9}(X)=(X-X^2)/2$
	g_{-8}	-1	0	1	$g_{-8}(X)=X$
	g_{-7}	-1	1	-1	$g_{-7}(X)=1-2X^2$
	g_{-6}	-1	1	0	$g_{-6}(X)=-1+(3X^2-X)/2$
	g_{-5}	-1	1	1	$g_{-5}(X)=1+X-X^2$
	g_{-4}	0	-1	-1	$g_{-4}(X)=-1+(X^2-X)/2$
	g_{-3}	0	-1	0	$g_{-3}(X)=X^2-1$
	g_{-2}	0	-1	1	$g_{-2}(X)=-1+(3X^2+X)/2$
	g_{-1}	0	0	-1	$g_{-1}(X)=-(X^2+X)/2$
	g_0	0	0	0	$g_0(X)=0$
	g_1	0	0	1	$g_1(X)=(X^2+X)/2$
	g_2	0	1	-1	$g_2(X)=1-(3X^2+X)/2$
	g_3	0	1	0	$g_3(X)=1-X^2$
	g_4	0	1	1	$g_4(X)=1-(X^2-X)/2$
	g_5	1	-1	-1	$g_5(X)=1-X+X^2$
	g_6	1	-1	0	$g_6(X)=1-(3X^2-X)/2$
	g_7	1	-1	1	$g_7(X)=2X^2-1$
	g_8	1	0	-1	$g_8(X)=-X$
	g_9	1	0	0	$g_9(X)=(X^2-X)/2$
	g_{10}	1	0	1	$g_{10}(X)=X^2$
	g_{11}	1	1	-1	$g_{11}(X)=1-X-X^2$
	g_{12}	1	1	0	$g_{12}(X)=1-(X+X^2)/2$
	g_{13}	1	1	1	$g_{13}(X)=1$

The other discrete functions are represented by a combination of these three functions in the following form,

$$g_j(X)=a_1^j g_1(X)+a_2^j g_2(X)+a_3^j g_3(X), \quad (j=-13,\,-12,\,\cdots,\,0,\,\cdots,\,12,\,13), \qquad (3.2.1)$$

where the coefficients a_1^j, a_2^j, and a_3^j respectively take appropriate one of the integers $\{-1, 0, 1\}$. These functions are identical to the point functions mentioned in the previous section. The point functions required are three orthogonal ones $\{I_1, I_2, I_3\}$ in a set of three integers. When we consider the logic NOT for the set $\{-1, 0, 1\}$, the definition of the Boolean algebra becomes meaningless. For this case, it is better to define the logic NOT by saying that there exists a pair concept of elements where 'NOT' indicates picking the opposite element of the pair. If we choose two functions in discrete functions on $\{-1, 0, 1\}$ suitably, then we can keep the definition of logic 'NOT' in the Boolean algebra. An example for a pair of discrete functions realizing the Boolean definition of logic NOT is $X+Y-XY+(1-X^2)(1-Y^2)$ and $X^2Y^2+(1-X^2)(1-Y^2)$. The former is the modification of logic OR, but latter is different from the logic 'AND'. Both have the same modification term $(1-X^2)(1-Y^2)$. Now we organize the recurrence formula to obtain the two-variable discrete functions on $\{-1, 0, 1\}$. The procedure is similar to Eqs (3.1.1) and (3.1.2). The recurrence equation for two-variable discrete functions $\{f_{ijk}\}$ is:

$$\begin{aligned} f_{ijk}(X,\,Y)&=1/2(X^2-X)g_i(Y)+(1-X^2)g_j(Y)+1/2(X^2+X)g_k(Y)\\ &=g_9(X)g_i(Y)+g_3(X)g_j(Y)+g_1(X)g_k(Y),\\ &\qquad (i,j,k\in\{-13,\,-12,\,\cdots,\,-1,\,0,\,1,\,\cdots,\,12,\,13\}). \qquad (3.2.2)\end{aligned}$$

The latter form is kept on the discrete functions of the set $\{0, 1, 2\}$ shown in Table 6.

We discuss the specificity of using the integers -1, 0, and 1. These integer satisfy the relations $X^2=X$ for the set $\{0, 1\}$ and $X^3=X$ for the set $\{-1, 0, 1\}$ or $\{-1, 1\}$, where X denotes an arbitrary element of each set. This characteristic of these values reduces higher-order powers of X to the lowest possible ones. But other choices of integer sets permit any type of functions using higher order power of X. An example is organized on the set $\{0, 1, 2\}$ (see Table 6). We show the results using the quadratic and cubic forms of the functions. This means any order of powers of the real functions can be used to organize a group of discrete functions, but the form of real functions does not have any meaning concerned with mathematics. The choice of the form of function is determined by the physical meaning of that choice. In mathematics, the following lemma has a significant meaning:

Lemma 3−2 (*Recurrence form of three-variable functions on three elements set*)

Let X, Y, and Z be elements on the set of three integers. There exist 19683($=3^w$, $w=3^2$) two-variable discrete functions, denoted by $\{f_0, f_1, \ldots, f_{19682}\}$ and three orthogonal functions of $\{I_0, I_1, I_2\}$, which are point functions of one-variable, like the Kronecker delta. Then all the discrete functions ($19683^3=3^w$, $w=3^3$) of three variables are completely

Table 6 One-variable functions on {0, 1, 2}

X		0 1 2	Quadratic form	Cubic form
	g_0	0 0 0	$g_0(X)=0$	0
	g_1	0 0 1	$g_1(X)=(X^2-X)/2$	$(X^3-X)/6$
	g_2	0 0 2	$g_2(X)=X^2-X$	$(X^3-X)/3$
	g_3	0 1 0	$g_3(X)=-X^2+2X$	$-(X^3-4X)/3$
	g_4	0 1 1	$g_4(X)=-(X^2-3X)/2$	$-(X^3-7X)/6$
	g_5	0 1 2	$g_5(X)=X$	X
	g_6	0 2 0	$g_6(X)=-2X^2+4X$	$-2(X^3-4X)/3$
	g_7	0 2 1	$g_7(X)=-(3X^2-7X)/2$	$-(X^3-5X)/2$
	g_8	0 2 2	$g_8(X)=-X^2+3X$	$-(X^3-7X)/3$
	g_9	1 0 0	$g_9(X)=1+(X^2-3X)/2$	$1+(X^3-7X)/6$
	g_{10}	1 0 1	$g_{10}(X)=1+X^2-2X$	$1+(X^3-4X)/3$
	g_{11}	1 0 2	$g_{11}(X)=1+(3X^2-5X)/2$	$1+(X^3-3X)/2$
	g_{12}	1 1 0	$g_{12}(X)=1-(X^2-X)/2$	$1-(X^3-X)/6$
$g(X)$	g_{13}	1 1 1	$g_{13}(X)=1$	1
	g_{14}	1 1 2	$g_{14}(X)=1+(X^2-X)/2$	$1+(X^3-X)/6$
	g_{15}	1 2 0	$g_{15}(X)=1-(3X^2-5X)/2$	$1-(X^3-3X)/2$
	g_{16}	1 2 1	$g_{16}(X)=1-(X^2-2X)1-(X^3-4X)/3$	
	g_{17}	1 2 2	$g_{17}(X)=1-(X^2-3X)/2$	$1-(X^3-7X)/6$
	g_{18}	2 0 0	$g_{18}(X)=2+X^2-3X$	$2+(X^3-7X)/3$
	g_{19}	2 0 1	$g_{19}(X)=2+(3X^2-7X)/2$	$2+(X^3-5X)/2$
	g_{20}	2 0 2	$g_{20}(X)=2+2(X^2-2X)$	$2+2(X^3-4X)/3$
	g_{21}	2 1 0	$g_{21}(X)=2-X$	$2-X$
	g_{22}	2 1 1	$g_{22}(X)=2+(X^2-3X)/2$	$2+(X^3-7X)/6$
	g_{23}	2 1 2	$g_{23}(X)=2+X^2-2X$	$2+(X^3-4X)/3$
	g_{24}	2 2 0	$g_{24}(X)=2+X^2-X$	$2-(X^3-X)/3$
	g_{25}	2 2 1	$g_{25}(X)=2-(X^2-X)/2$	$2-(X^3-X)/6$
	g_{26}	2 2 2	$g_{26}(X)=2$	2

described by the following recurrence equation,

$$F_{ijk}(X, Y, Z)=I_0(X)f_i(Y, Z)+I_1(X)f_j(Y, Z)+I_2(X)f_k(Y, Z),$$

$$(i, j, k \in \{0, 1, 2, \cdots, 19682\}). \quad (3.2.3)$$

[*Proof*] Two-variable functions produced by the equation above are independent of each other. Each point function is orthogonal to the other. This means that

$$I_0(X)I_1(X)=I_1(X)I_2(X)=I_2(X)I_0(X)=0,$$

and also means each orthogonal function takes the value 1 at the specified value. These are three for the three-element set. Any triple combination of independent functions determines the whole discrete function space of three variables X, Y, and Z. [*End of proof*]

3.3 Generalized Description of Discrete Functions

We now generalize the results obtained in subsections 3.1 and 3.2. When we apply mathematical induction to the results of the previous subsections, the following lemmae are established.

Lemma 3-3 (*Recurrence form of n-variable functions on two elements set*)
Let X_1, X_2, ..., and X_{n+1} be elements on the set of two integers, and let N be the number of all n-variable discrete functions on the set. There exist $N\,(=2^w,\ w=2^n)$ n-variable discrete functions denoted by $\{f_0, f_1, ..., f_{N-1}\}$, and two orthogonal functions of $\{I_0, I_1\}$ which are point functions of one variable, like the Kronecker delta. Then all N^2 discrete functions of $n+1$ variables are completely described by the following recurrence equation,

$$F_{ij}(X_1, X_2, \cdots, X_n, X_{n+1}) = I_0(X_{n+1}) f_i(X_1, X_2, \cdots, X_n) + I_1(X_{n+1}) f_j(X_1, X_2, \cdots, X_n),$$
$$(i, j \in \{0, 1, \cdots, N-1\}). \quad (3.3.1)$$

Lemma 3-4 (*Recurrence form of n-variable functions on three elements set*)
Let X_1, X_2, ..., and X_{n+1} be elements on the set of three integers, and let N be the number of all n-variable discrete functions on the set. There exist $N\,(=3^w,\ w=3^n)$ n-variable discrete functions denoted by $\{f_0, f_1, ..., f_{N-1}\}$, and three orthogonal functions of $\{I_0, I_1, I_2\}$ which are point functions of one variable, like the Kronecker delta. Then all N^3 discrete functions of $n+1$ variables are completely described by the following recurrence equation,

$$F_{ijk}(X_1, X_2, \cdots, X_n, X_{n+1}) = I_0(X_{n+1}) f_i(X_1, X_2, \cdots, X_n) + I_1(X_{n+1}) f_j(X_1, X_2, \cdots, X_n)$$
$$+ I_2(X_{n+1}) f_k(X_1, X_2, \cdots, X_n), \qquad (i, j, k \in \{0, 1, \cdots, N-1\}). \quad (3.3.2)$$

The above considerations allow us the following theorem:

Theorem 1 (*Recurrence form of n-variable functions on m-element set*)
Let X_1, X_2, ..., and X_{n+1} be elements on the set of m integers, and let N be the number of all n-variable discrete functions on the set. There exist $N\,(=m^w,\ w=m^n)$ n-variable discrete functions denoted by $\{f_0, f_1, ..., f_{N-1}\}$, and m orthogonal functions of $\{I_0, I_1, I_2, ..., I_{m-1}\}$ which are point functions of one variable, like the Kronecker delta. Then all N^m discrete functions of $n+1$ variables are completely described by the following recurrence equation,

$$F_{ijk \cdots l}(X_1, X_2, \cdots, X_n, X_{n+1}) = I_0(X_{n+1}) f_i(X_1, X_2, \cdots, X_n) + I_1(X_{n+1}) f_j(X_1, X_2, \cdots, X_n)$$
$$+ I_2(X_{n+1}) f_k(X_1, X_2, \cdots, X_n) + \cdots + I_{m-1}(X_{n+1}) f_l(X_1, X_2, \cdots, X_n),$$
$$(i, j, k, \cdots, l \in \{0, 1, \cdots, N-1\}), \quad (3.3.3)$$

where l signifies m-th suffix.

[*Proof*] There exist m^m one-variable discrete functions on the m elements set. The m orthogonal functions can be constructed by a combination of these distinct functions. Two-variable discrete functions be generated by use of m orthogonal functions and m^m one-variable functions. The recurrence form of three variable functions can be organized similarly to lemmae 3–1 and 3–2. Mathematical induction leads the recurrence form between n and $n+1$ variables. [*End of proof*]

4. Applications of Discrete Algebra to Cellular Automata and Binary Textures

One rationale for studying cellular automata was to describe soliton-like behaviour [1]. A second objective of our research has been the study of human discrimination of binary textures [3]. Both studies involve aspects of cellular automata. Usually three-variable functions are used for these studies, so that the general description of three-variable discrete functions is very convenient for us. The lemmae 3–1 and 3–2, and equations (3.1.1), (3.1.2), (3.2.1), and (3.2.2) give the following form of three variable discrete functions:
For the set $\{0, 1\}$,

$$\begin{aligned}F(X, Y, Z)=&A_1(1-X)(1-Y)(1-Z)+A_2(1-X)(1-Y)Z+A_3(1-X)Y(1-Z)\\&+A_4(1-X)YZ+A_5X(1-Y)(1-Z)+A_6X(1-Y)Z\\&+A_7X(1-Y)Z+A_8XYZ,\\&\qquad\qquad A_i\in\{0, 1\}, (i=1, 2, \cdots, 8),\end{aligned}\tag{4.1}$$

for the set $\{-1, 1\}$,

$$\begin{aligned}F(X, Y, Z)=&A_1(1-X)(1-Y)(1-Z)/8+A_2(1-X)(1-Y)(1+Z)/8\\&+A_3(1-X)(1+Y)(1-Z)/8+A_4(1-X)(1+Y)(1+Z)/8\\&+A_5(1+X)(1-Y)(1-Z)/8+A_6(1+X)(1-Y)(1+Z)/8\\&+A_7(1+X)(1+Y)(1-Z)/8+A_8X(1+X)(1+Y)(1+Z)/8,\\&\qquad\qquad A_i\in\{-1, 1\}, (i=1, 2, \cdots, 8),\end{aligned}\tag{4.2}$$

and for the set $\{-1, 0, 1\}$,

$$\begin{aligned}F(X, Y, Z)=&A_1(X^2-X)(Y^2-Y)(Z^2-Z)/8+A_2(X^2-X)(Y^2-Y)(1-Z^2)/4\\&+A_3(X^2-X)(Y^2-Y)(Z^2+Z)/8+A_4(X^2-X)(1-Y^2)(Z^2-Z)/4\\&+A_5(X^2-X)(1-Y^2)(1-Z^2)/2+A_6(X^2-X)(1-Y^2)(Z^2+Z)/4\\&+A_7(X^2-X)(Y^2+Y)(Z^2-Z)/8+A_8(X^2-X)(Y^2+Y)(1-Z^2)/4\\&+A_9(X^2-X)(Y^2+Y)(Z^2+Z)/8+A_{10}(1-X^2)(Y^2-Y)(Z^2-Z)/4\\&+A_{11}(1-X^2)(Y^2-Y)(1-Z^2)/2+A_{12}(1-X^2)(Y^2-Y)(Z^2+Z)/4\\&+A_{13}(1-X^2)(1-Y^2)(Z^2-Z)/2+A_{14}(1-X^2)(1-Y^2)(1-Z^2)\\&+A_{15}(1-X^2)(1-Y^2)(Z^2+Z)/2+A_{16}(1-X^2)(Y^2+Y)(Z^2-Z)/4\end{aligned}$$

$$+A_{17}(1-X^2)(Y^2+Y)(1-Z^2)/2+A_{18}(1-X^2)(Y^2+Y)(Z^2+Z)/4$$
$$+A_{19}(X^2+X)(Y^2-Y)(Z^2-Z)/8+A_{20}(X^2+X)(Y^2-Y)(1-Z^2)/4$$
$$+A_{21}(X^2+X)(Y^2-Y)(Z^2+Z)/8+A_{22}(X^2+X)(1-Y^2)(Z^2-Z)/4$$
$$+A_{23}(X^2+X)(1-Y^2)(1-Z^2)/2+A_{24}(X^2+X)(1-Y^2)(Z^2+Z)/4$$
$$+A_{25}(X^2+X)(Y^2+Y)(Z^2-Z)/8+A_{26}(X^2+X)(Y^2+Y)(1-Z^2)/4$$
$$+A_{27}(X^2+X)(Y^2+Y)(Z^2+Z)/8,$$
$$A_i \in \{-1, 0, 1\}, \quad (i=1, 2, \cdots, 27) \qquad (4.3)$$

These forms include all possible three-variable discrete functions, as we can change the permitted coefficients, $A_1, A_2, \ldots, A_K$ ($K=8$ for two integers and $K=27$ for three integers), in any integer of the set. We introduce a coefficient vector A, that has elements which are coefficients, $A_1, A_2, \ldots, A_K$ ($K=8$ for two integers and $K=27$ for three integers). For example, for the two-element set, this vector A is

$$A=(A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8), \qquad (4.4)$$

and for three-element set, it is

$$A=(A_1, A_2, A_3, \ldots, A_{25}, A_{26}, A_{27}). \qquad (4.5)$$

A coefficient vector A is identical to a rule of cellular automata, because each factor is a point function which take the value 1 at specified value of (X, Y, Z). For the set $\{0, 1\}$, an assignment of A to an integer gives Wolfram's rule number [6].

In a cellular automaton, the dynamics of the state of whole elements can be analyzed. The local state of elements produces an evolving temporal pattern of the overall state of the elements, *viz. patten dynamics*. For the three-neighbour case, the temporal development of each cell state can be described by the following abstract form:

$$S(t+1)=F(S'(t), S^*(t), S''(t)), \qquad (4.6)$$

where $S(t)$ means the state of a certain cell in the system at time t, superscripts signify the difference of specified cells, and F is a function defining how to develop the cell states temporally. We call F a rule, and this is equivalent to one of three-variable discrete functions discussed here. The rule F can be expressed as

$$F=A\cdot \boldsymbol{P}, \qquad (4.7)$$

where A is the coefficient vector discussed above, and $\boldsymbol{P}$ is the point function vector whose elements are triple products of orthogonal functions. The explicit expression of $\boldsymbol{P}$ takes the following form:

$$\boldsymbol{P}=(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8), \quad \text{for two-element set}, \qquad (4.8)$$
$$\boldsymbol{P}=(p_1, p_2, p_3, \ldots, p_{25}, p_{26}, p_{27}), \quad \text{for three-element set}, \qquad (4.9)$$

where each element of $\boldsymbol{P}$ in sets of two integers (for (4.8)) is defined as follows,

$$
\begin{aligned}
&p_1=I_0(X)I_0(Y)I_0(Z), \quad p_2=I_0(X)I_0(Y)I_1(Z)\\
&p_3=I_0(X)I_1(Y)I_0(Z), \quad p_4=I_0(X)I_1(Y)I_1(Z)\\
&p_5=I_1(X)I_0(Y)I_0(Z), \quad p_6=I_1(X)I_0(Y)I_1(Z)\\
&p_7=I_1(X)I_1(Y)I_0(Z), \quad p_8=I_1(X)I_1(Y)I_1(Z),
\end{aligned} \tag{4.10}
$$

and elements of $\boldsymbol{P}$ in sets of three integers (for (4.9)) are

$$
\begin{aligned}
&p_1=I_0(X)I_0(Y)I_0(Z), \quad p_2=I_0(X)I_0(Y)I_1(Z), \quad p_3=I_0(X)I_0(Y)I_2(Z)\\
&p_4=I_0(X)I_1(Y)I_0(Z), \quad p_5=I_0(X)I_1(Y)I_1(Z), \quad p_6=I_0(X)I_1(Y)I_2(Z)\\
&p_7=I_0(X)I_2(Y)I_0(Z), \quad p_8=I_0(X)I_2(Y)I_1(Z), \quad p_9=I_0(X)I_2(Y)I_2(Z)\\
&p_{10}=I_1(X)I_0(Y)I_0(Z), \quad p_{11}=I_1(X)I_0(Y)I_1(Z), \quad p_{12}=I_1(X)I_0(Y)I_2(Z)\\
&p_{13}=I_1(X)I_1(Y)I_0(Z), \quad p_{14}=I_1(X)I_1(Y)I_1(Z), \quad p_{15}=I_1(X)I_1(Y)I_2(Z)\\
&p_{16}=I_1(X)I_2(Y)I_0(Z), \quad p_{17}=I_1(X)I_2(Y)I_1(Z), \quad p_{18}=I_1(X)I_2(Y)I_2(Z)\\
&p_{19}=I_2(X)I_0(Y)I_0(Z), \quad p_{20}=I_2(X)I_0(Y)I_1(Z), \quad p_{21}=I_2(X)I_0(Y)I_2(Z)\\
&p_{22}=I_2(X)I_1(Y)I_0(Z), \quad p_{23}=I_2(X)I_1(Y)I_1(Z), \quad p_{24}=I_2(X)I_1(Y)I_2(Z)\\
&p_{25}=I_2(X)I_2(Y)I_0(Z), \quad p_{26}=I_2(X)I_2(Y)I_1(Z), \quad p_{27}=I_2(X)I_2(Y)I_2(Z),
\end{aligned} \tag{4.11}
$$

respectively. Since $\boldsymbol{P}$ is fixed, F varies due to the change in $\boldsymbol{A}$. Both F and $\boldsymbol{A}$ refer to the rule of cellular automata as far as $\boldsymbol{P}$ is the vector consisting of point functions. The expression (4.7) is quite similar to the expression obtained from matrix approach to the rules [1]. The point function vector $\boldsymbol{P}$ is identical to νS^{-1} of ref. [1]. Rewriting each element of νS^{-1} shows a clear correspondence to the element of $\boldsymbol{P}$. When the vector $\boldsymbol{A}$ is changed by a quantity which reflects the state of system, the system has a temporal change of rules which called 'rule dynamics' [7].

Binary textures, which are used in the study of human texture discrimination, consist of patterns of square pixels. Each pixel is coloured white or black. In such a texture, white and black signify two states. The assignment of white or black to two integers is arbitrary. It is possible to use either $\{0, 1\}$ or $\{-1, 1\}$. The isotrigon textures, sets of textures having identical average third-order (triple) correlation functions [8], are of particular interest, because human can easily discriminate collections of such textures, thus indicating that we exploit information about higher-order ($\geqq 4^{\text{th}}$) correlations within images. Primate cortical neurons have been shown to be sensitive to these higher-order correlations [9]. The isotrigon textures used to date in studies of texture vision are made by a recursion rule employing triple products of pixels [3, 10] using the set of integers $\{-1, 1\}$. The use of $\{-1, 1\}$sets the average brightness of the texture at zero. This is suitable when considering real visual systems that encode images in terms of positive and negative contrasts.

The discrete functions described above indicate the existence of isotrigon textures other than those used to date [3, 10]. For example, patterns containing pixels with the intermediate brightness (gray), together with black and white pixels, can immediately be assigned to $\{-1, 0, 1\}$. These textures can be obtained by using the discrete functions described above for three-integer sets. Note that the three-variable discrete functions include the one- and two-variable discrete functions less than N. In the n-variable case, we can consider any physiological expectations for texture discrimination mechanisms. Thus, apart from mathematical interest, the present study opens the way to provide a wider range of stimuli that are potentially useful for the study of human texture discrimination mechanisms.

References

[1] A. Ankiewicz and Y. Nagai, ''Soliton-type effects in three-level cellular automata''. *Mem. Kokushikan Univ. CIS*, **22** (2001) 1–9.
[2] Y. Nagai and T. Maddess, ''Randomness on Binary Textures in Vision Research'', *AIP Proceedings*, No. **519**, (2000) 501–503.
[3] T. Maddess and Y. Nagai, ''Discrimination of Isotrigon Textures'', Submitted to Vision Research.
[4] J. Eldon Whitesitt, Boolean Algebra and Its Applications, Dover, New York, 1995.
[5] K. Tezuka, The foundations of electrical computation equipments (in Japanese), Shoko-Do, Tokyo, 1978.
[6] S. Wolfram, Theory and Application of Cellular Automata, World Scientific, Singapore, 1986.
[7] Y. Nagai and Y. Aizawa, ''Ruledynamical generalization of McCulloch-Pitts neuron network''. *BioSystems*, **58**, (2000) 177–185.
[8] J. D. Victor, ''Images, statistics and textures: implications of triple correlation uniqueness for texture statistics and Julez conjecture: comment''. *J. Opt. Soc. Am.*, **A11**, (1994) 1680–1684.
[9] K. P. Purpura, J. D. Victor, and E. Katz, ''Striate cortex extracts higher-order spatial correlations from visual textures''. *Proc. Natl. Acad. Sci. U S A*, **91**, (1994) 8482–8486.
[10] J. D. Victor and M. M. Conte, ''Spatial Organization of Nonlinear Interactions in Form Reception''. *Vision Research*, **31**, (1991) 1457–1488.