Paper

Soliton-type effects in three-level cellular automata

ADRIAN ANKIEWICZ[1] and YOSHINORI NAGAI[2,3]

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Synopsis: We show that soliton-type effects, such as fusion, birth of new solitons, and elastic collisions, are possible for some cellular automata with quite simple nonlinear evolution rules.

1 Introduction

Various important physical effects are described by nonlinear equations. For example, optical fibre pulse propagation is found from the nonlinear Schrödinger equation (NLSE). In this case, bright solitons are formed because the nonlinear self-focussing counteracts the dispersion or diffraction of the light. Although partial differential equations (p.d.e's) are used for most physical applications, cellular automata (CA) provide the possibility of an different viewpoint [1] for simulating physical behaviour. The alternative nature of this approach has been stressed in [2]. Cluster formation, with reference to biological systems is studied in [3], while the use of reversible automata for statistical mechanics is considered in [4]. CAs can model the motion of bacteria as they seek to congregate [5].

In [6], T. Tokihiro et al. use a 2–valued filter CA which requires counting the number of 'ones' from the left in order to determine the sequence evolution. In their case, an isolated sequence of n ones travels to the right with velocity n. Due to the different speeds, collisions are possible, and then each sequence continues at its original velocity, n, but it has experienced a 'phase shift'. This behaviour resembles solitons of the KdV equation, where soliton velocity depends on amplitude, so that high solitons can overtake low ones, also producing a 'phase shift'. The authors of [6] relate their filter CA to the KdV through the Lotka-Volterra equation, an integrable discretization of the KdV. This work is important in that it links CAs with p.d.e's through a limiting procedure.

However, it is probably more realistic physically to use local rules rather than filter CAs, since solitons of p.d.e's represent localized effects.

We mention the nonlinear Schrödinger equation, with a generalized nonlinearity law, as a commonly used model:

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[1] Optical Sciences Centre, [2] Applied Mathematics Dept.; Australian National University, Canberra, ACT 0200.

[3] Centre for Information Science and School of Political and Economic Sciences, Kokushikan University, 4–28–1 Setagaya, Tokyo, 154–8515, Japan.

$$i\psi_z + \frac{1}{2}\psi_{tt} + \psi N(|\psi|^2) = 0. \tag{1}$$

This equation, and others related to it, have been discussed [7]. When applied to propagation in optical fibers, ψ is the field, z is the (normalized) distance along the fiber and t is the retarded time (meaning that the reference frame is moving at the pulse group velocity). On the other hand, for spatial propagation in planar structures, z is the (normalized) distance along the waveguide, while t is the transverse dimension. Thus the light intensity is $|\psi|^2$, and the form of nonlinearity means that the function N depends on the intensity only. If N is the identity function, then we have the common Kerr-law behaviour and a mathematically integrable system which supports solitons. In this case, solitons can pass through each other and suffer only a phase shift and no energy radiation loss. The exact mathematical description of this process, together with a proposal of using such an 'X-junction' as an optical switch, was given in [8].

A soliton here is formed when the nonlinearity cancels the dispersion. If we take the Kerr-law form for the nonlinearity, then the exact pulse shape required is

$$\psi = f \exp(iq^2 z/2)$$

where $f = q\,sech(qt)$ for any value of q. For this required shape, if we divide eqn. (2) through by ψ, we see that the dispersion (diffraction for spatial case) term,

$$\frac{\psi_{tt}}{(2\psi)} = q^2\left(\frac{1}{2} - \mathrm{sech}\,(qt)^2\right)$$

is cancelled by the evolution term, $i\psi_z/\psi = -q^2/2$ acting with the nonlinearity $|\psi|^2 = q^2 sech^2(qt)$.

All real materials saturate if the intensity is sufficiently high, so to study non-elastic collisions of soltions, it is convenient to use a form of N which includes saturation, e.g.

$$N = \frac{|\psi|^2}{1 + \gamma|\psi|^2} \tag{2}$$

Thus at low intensity, the index increases linearly with intensity, while at high intensities, it approaches a constant. Solitons in saturable materials have been studied using p.d.e's in [9]–[11]. For any form of nonlinearity other than the Kerr-law noted above, the solitons do not pass through each other unscathed.

2 Nonlinear discrete dynamics

We seek to present discrete cellular automata which demonstrate some of the above features from the continuous equations.

We now present our simplest model which demonstrates some of the observed effects, such as soliton fusion, using a nonlinearity in the cellular automata rule. The rule is defined as follows. The element $m(j, k)$, which is the kth element in row j, is determined by the element k in row $j-1$, *viz.* $m(j-1, k)$, and the 2 neighbours of that element, *viz.* $m(j-1, k-1)$ and $m(j-1, k+1)$. We have

$$m(j, k)=m(j-1, k)+m^2(j-1, k)+Floor \times\left(\frac{1}{2}[m(j-1, k-1)+m(j-1, k+1)]-m(j-1, k)\right), \tag{3}$$

for $j=2, 3, \cdots$. The term $m^2(j-1, k)$ is the nonlinear one. It would be analogous to $|\psi|^2$ in the continuous case. The function $Floor(x)$ indicates the largest integer less than or equal to x. The term including this function is analogous to a second derivative (*viz.* the dispersion or diffraction in optical propagation) and has the effect of making a cell value 'closer' to the values of its neighbours.

The input sequence ($j=1$) assumes a localized or 'pulse-like' form. Thus we need sufficient zeros to the left and right of the 'pulse'. As with solitons from differential equations, the nonlinearity here acts as a 'glue' holding some pulses together.

2.1 Solitons

A soliton is a sequence which maintains its form, perhaps with lateral translation, on evolution. With this rule, we find that

$$(\cdots 0, 0, 0, 1, 0, 0, 0, \cdots)$$

is a soliton. This occurs because this pattern remains invariant on iteration. This is also true for any collections of 'ones' separated by at least 2 'zeros', eg.

$$(0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, \cdots)$$

Another interesting soliton is

$$(0, 0, 0, 1, 1, 0, 0, 0, 0, \cdots) \tag{4}$$

The basic soliton unit which moves to the right with unit velocity, which we label 'R', is

$$(0, 0, 0, 1, -1, 0, 0, 0, 0, \cdots)$$

Thus, its evolution is given by

$$\begin{pmatrix} \cdots & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & \cdots \end{pmatrix}, \tag{5}$$

etc. A NLSE soliton has a fixed phase across it. If we describe the above cell soliton as having zero phase, then the one with a phase of π is its reverse, and travels in the opposite direction. It has velocity of -1, and we label it 'L':

$$\begin{pmatrix} \cdots & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}, \tag{6}$$

etc.

2.2 Soliton fusion

Demonstrations of the fusion of solitons and the 'birth' of new solitons have been given in [12] and [13]. This effects indicate considerable inelasticity, showing that the differential equation representing the system is far from integrable.

We find that our model system can describe this fusion situation. Fusion occurs when a left-moving soliton collides with a right-moving one. To achieve this, we intially need to have at least 2 zeros between the element solitons:

$$\begin{pmatrix} \cdots & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}. \tag{7}$$

Thus the 2 solitons have fused to form a single soliton of the form, of (4) above, and this 'propagates' forever.

2.3 Birth of solitons

Our rule is also capable of showing how a single linked pulse can give rise to the birth of new solitons. Here is an example of it:

$$\begin{pmatrix} \cdots & 0 & 0 & 1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \end{pmatrix}. \tag{8}$$

Thus the two individual 'daughter' solitons then continue forever. In physical systems 2 solitons will often continue separately or fuse, depending on their relative phases, as in our examples. This can be seen, for example, in fig. 2, parts (a) and (b) of the experiment in [13].

3 Soliton effect 3–level system

We now construct a system where each element takes on one of three levels (here -1, 0 and $+1$). As in the system described in the earlier section, each element depends on its value and that of its 2 neighbours in the previous stage. For conveience, we write $a=m(j-1, k-1)$, $b=m(j-1, k)$ and $c=m(j-1, k+1)$. The input (0, 0, 0) must produce a zero output for the new centre cell to ensure localization, so we need only $3^3-1=26$ linearly-independent combinations of (a, b, c) to provide all the required coefficients.

Thus we write

$$\begin{aligned}f=&ax_1+bx_2+cx_3+a^2x_4+b^2x_5+c^2x_6+abx_7+acx_8+bcx_9\\&+a^2bx_{10}+a^2cx_{11}+ab^2x_{12}+b^2cx_{13}+ac^2x_{14}+bc^2x_{15}+a^2b^2x_{16}+a^2c^2x_{17}+b^2c^2x_{18}+abcx_{19}\\&+a^2b^2cx_{20}+ab^2c^2x_{21}+a^2bc^2x_{22}+a^2b^2c^2x_{23}+abc^2x_{24}+a^2bcx_{25}+ab^2cx_{26}.\end{aligned}\tag{9}$$

In other words, we have

$$f=\mathbf{v}\cdot\mathbf{x}$$

where $\mathbf{v}$ is the vector formed from (the 26) coefficients of the x's,

$$\mathbf{v}(a, b, c)=(a, b, c, a^2, \cdots, ab^2c),$$

while $\mathbf{x}$ is the vector

$$\mathbf{x}=(x_1, x_2, x_3, \cdots, x_{26}).$$

As noted above, no constant term is required in this form. Now, let p_i be the value required for the ith mapping, when the ith value of (a, b, c) is substituted into eqn. (9), where the triplets (a, b, c) are arranged starting from $p_1=f(-1, -1, -1)$, $p_2=f(-1, -1, 0)$, etc., and ending up with $p_{26}=f(1, 1, 1)$. Thus each p_i is equal to -1 or 0 or 1. The (26×26) matrix S is formed form the elements $\mathbf{v}$ as follows:

$$S=\begin{pmatrix}\mathbf{v}(-1, -1, -1\\ \mathbf{v}(-1, -1, 0)\\ \mathbf{v}(-1, -1, 1)\\ \cdots\\ \mathbf{v}(0, 0, -1)\\ \mathbf{v}(0, 0, 1)\\ \cdots\\ \mathbf{v}(1, 1, 0)\\ \mathbf{v}(1, 1, 1)\end{pmatrix}.\tag{10}$$

Thus $S\mathbf{x}=\mathbf{p}$. We choose convenient values of the elements of vector $\mathbf{p}$ from the numbers $\{-1, 0, 1\}$.

We can obtain interesting physical effects by choosing the vector $\mathbf{p}$ in a suitable way. We note again that $f(0, 0, 0)=0$, so $f(0, 0, 0)$ is excluded from the above numbering, as seen in the middle of the matrix (10).

We thus have

$$\mathbf{x}=S^{-1}\mathbf{p}$$

and the inverse of matrix S, *viz*. S^{-1} is given in Table 1.

$$\left(\begin{array}{cccccccccccccccccccccccccc}
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & -\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 & 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & 0 & -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & -\frac{1}{8} & 0 & 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} \\
-\frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{8} & 0 & \frac{1}{8} \\
-\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \\
-\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \\
\frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \\
\frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & 0 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \\
\frac{1}{8} & 0 & -\frac{1}{8} & 0 & 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{8} & 0 & -\frac{1}{8} & 0 & 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} \\
\frac{1}{8} & 0 & -\frac{1}{8} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{8} & 0 & -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{8} & 0 & \frac{1}{8}
\end{array}\right)$$

TABLE 1. Inverse of matrix S.

3.1 Period–2 solitons

To get the soliton effects, we use the following form for the vector $\mathbf{p}$:

$$\mathbf{p}=(0, 0, 1, -1, -1, 0, 0, 0, 1, 0, 1, 1, -1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0) \tag{11}$$

As noted above, by inverting the matrix S, we find the required vector $\mathbf{x}$, i.e.:

$$\mathbf{x}=\left(\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -\frac{1}{2}, -\frac{1}{2}, 0, -\frac{1}{2}, 0, 0, -\frac{1}{2}\right) \tag{12}$$

We can simplify eq. (9), using eq. (12) to get the expression for the new centre element:

$$f=\frac{1}{2}(a+c)-\left[\frac{1}{2}(a^2+c^2)-b^2\right]+a^2c^2-\frac{1}{2}ab^2c(a+c+ac+1), \tag{13}$$

i.e.

$$f=\frac{1}{2}a(1-a)+\frac{1}{2}c(1-c)+b^2+a^2c^2-\frac{1}{2}ab^2c(a+c+\mathrm{ac}+1).$$

The term $1/2(a^2+c^2)-b^2$ in eqn. (13) corresponds to a second derivative of intensity term. We note that f depends only on b^2 and not b alone. This is analogous to the nonlinearity depending only on intensity in the NLSE. For any combination of the 3 input values, this rule always produces a new sequence consisting of terms with these values also. It is not identical to the rule considered above, but it produces related soliton-type effects, and the advantage of this approach is that any rule can be programmed.

For example, we have soliton sequences $(\cdots 0, 1, 0\cdots)$ and $(\cdots 0, 1, 1, 0\cdots)$. A new pattern is the 'period-2 soliton'(see [7] for p.d.e. examples), which is a soliton which returns to its original form every second iteration. This can be started with the sequence $(\cdots 0, 0, 1, 1, 1, 0, 0\cdots)$, and it evolves to $(\cdots 0, 0, 1, 0, 1, 0, 0\cdots)$ and back to $(\cdots 0, 0, 1, 1, 1, 0, 0\cdots)$ etc.

We still retain the basic soliton unit moving to the right (R), *viz.*

$$(\cdots 0, 0, 0, 1, -1, 0, 0, 0, 0, \cdots),$$

and the left-moving soliton (L) *viz.*

$$(\cdots 0, 0, 0, -1, 1, 0, 0, 0, 0, \cdots).$$

If there are an even number of zeros between these 2 solitons, then they collide as before to fuse and produce the $(\cdots 0, 1, 1, 0\cdots)$ soliton.

If there are 4 or more isolated ones in a sequence, then we find the 'birth' of a 'daughter' soliton at each end of the sequence, e.g.

$$\begin{aligned}(\cdots 0, 0, 1, 1, 1, 1, 0, 0 \cdots) &\longrightarrow (\cdots 0, 0, 1, 0, 0, 1, 0, 0 \cdots)\\ &\longrightarrow (\cdots 0, 0, 1, 0, 0, 1, 0, 0 \cdots).\end{aligned}$$

3.2 Elastic collision

We now present a rule which provides for the well-known soliton elastic collision. This is the hallmark of an integrable system, and it indicates that no energy is lost in the collison. Here the L and R solitons pass through each other, suffering only a phase shift.

$$\mathbf{p}=(0, 0, 1, -1, -1, 0, 0, 0, 1, 0, 1, 1, -1, 0, 0, 1, 1, 1, 1, -1, 0, 0, 0, 1, 1, 0) \tag{14}$$

Again, inverting the matrix S gives us the required vector $\mathbf{x}$:

$$\mathbf{x}=\left(\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2}, 0, -\frac{1}{4} 0, 0, -\frac{1}{4}, 0, 0, -\frac{1}{4}, 0, 0, \frac{3}{4}, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{3}{8}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{8}, \frac{1}{8}, -\frac{3}{8}\right) \tag{15}$$

This produces

$$f=\frac{1}{2}(a+c)-\left[\frac{1}{2}(a^2+c^2)-b^2\right] + \frac{1}{8} ac[6ac-2-2a-2c+b(1+a-3b-3ab+c+ac-3bc-3abc)]. \tag{16}$$

Here the first term is the average of the neighbours while the second is a type of dispersion.

Here is the elastic collision. It requires an odd number of zeros between the L and R solitons intially.

$$\begin{pmatrix} \cdots 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \cdots \\ \cdots 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 \cdots \\ \cdots 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \cdots \\ \cdots 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \cdots \\ \cdots 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \cdots \\ \cdots 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 \cdots \\ \cdots 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \cdots \end{pmatrix}. \tag{17}$$

Thus the L soltion continues moving to the left after the collision, while the R soliton moves to the right. Each has suffered a unit shift in 'phase' relative to the position it would have had, if there was no collision.

If there are an even number of zeros between the L and R initial solitons, then we obtain fusion, as in the earlier cases. However, we do not obtain the breather solution here, since the sequence (0, 0, 1, 1, 1, 0, 0) evolves to (0, 0, 1, 0, 1, 0, 0) and remains as (0, 0, 1, 0, 1, 0, 0) forever.

4 Discussion

We now rewrite eqs. (13) and (16), and get the following expression which is useful in discussing the changes caused by the replacement of values corresponding to $p_{20}=f(1, -1, 1)$ from 0 to -1, and changing $p_{23}=f(1, 0, 1)$ from 1 to 0. The result is as follows:

$$f=b+\frac{1}{2}(a+c-2b)(1-acX(b))-\frac{1}{2}(a^2+c^2-2b^2)-abc-\mathrm{ac}Y(b)+a^2c^2Z(b)$$

For eq. (13), X, Y, and Z are $X=b^2$, $Y=1/2b^2$, $Z=1-1/2b^2$. These are invariant with respect to b changing sign. For eq. (16), we have

$$X=1+\frac{b}{2}-\frac{3}{2}b^2,$$

$$Y=1+\frac{b}{2}-\frac{1}{2}b^2,$$

$$Z=1+\frac{b}{6}-\frac{1}{2}b^2.$$

The main changes occur in $X(b)$ and $Y(b)$. In the case of eq. (13), $X(b)=1, 0, 1$ for $b=-1, 0, 1$, and $Y(b)=1/2, 0, 1/2$ for $b=-1, 0, 1$, respectively. On the other hand, the case of eq. (16) yields $X(b)=-1, 1, 0$ for $b=-1, 0, 1$ and $Y(b)=-1, 1, 1$ for $b=-1, 0, 1$, respectively.

5 Conclusion

We have demonstrated typecal soliton behaviour of non-integrable systems with very simple local rules with one or more nonlinear terms. These give some insight into the way in which even a simple nonlinearity can act to keep a pulse together or cause it to separate into distinct pulses, depending on the relative 'phase'. We have also demonstrated a 'lossless' soliton collision, with pulses passing through each other.

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