Paper

One-Dimensional Fractal Map

YOSHINORI NAGAI* AND YOJI AIZAWA**

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Abstract: One-dimensional fractal map (1-d fractal map) is consisting of submaps which arranges self-similar manner in the definition region of map. The 1-d fractal maps have discontinuity in their definition region. The orbit generated by recurrently applying of the 1-d fractal map for the mapping variable means the dynamics of 1-d fractal map. The complicated intermittent orbits are generated by the 1-d fractal M map. This implies chaotic itineracy of the 1-d fractal map.

1. Introduction

The idea of one-dimensional fractal map (1-d fractal map) was obtained by the investigation of neighborhood-three cellular automata with two states. We have studied the pattern dynamics of cellular automata in the view point of pattern-to-pattern mapping. In neighborhood-three cellular automata, pattern dynamics is governed by a local rule which determines the state of each cell at time t using the states of neighborhood-three cells around it and itself at time $t-1$. Although local rule is applied for whole cells array in the same way, the pattern of cells' states is different depending on their initial states. If one want to know the relation between whole cells' patterns at time t and $t+1$, it is suitable to introduce some scalar variable corresponding to the pattern of whole cells' states. For this purpose, we introduce rational number of binary cardinal which is corresponding to one-dimensional array of cells' states at time t defined as

$$\xi(t)=\sum_{i=1}^{n} 2^{-i}\mathrm{S}i(t)$$

where ξ denotes a rational number, S_i the state of i-th cell, n the number of cells composing the system of cellular automata, respectively. As seen from this definition, the correspondence between a rational number and a pattern of 1-d array of whole cells' states is one-to-one.

The mapping of pattern-to-pattern in cellular automata of neighborhood-three shows fractal nature. Wolfram touched on pattern-to-pattern mapping as a global nature of the pattern in his review article [1]. The mapping of pattern-to-pattern can be represented by utilizing the rational number $\xi(t)$ so that the mapping f: $\xi(t)\rightarrow\xi(t+1)$ is rational one. This

*Center for Information Science, and School of Political and Economic Sciences, Kokushikan University, 4-28-1 Setagaya, Tokyo 154, Japan and **Department of Applied Physics, School of Science and Engineering, Waseda University, 3-4-1 Nishi-Ookubo, Shinjyuku, Tokyo 169-50, Japan.

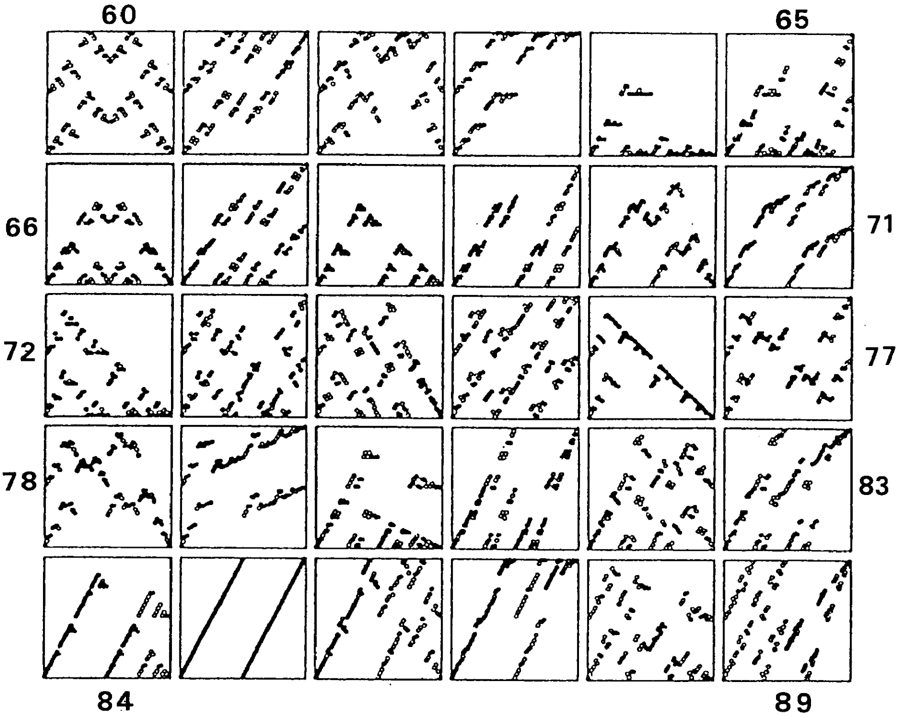

Fig. 1 Examples of Pattern-to-Pattern Mapping
Pattern-to-pattern mappings for Wolfram's rules (from 60 to 89) are shown. Global nature of pattern shows fractality.

mapping is a subset of rational numbers. Every point of mapping for pattern is isolated. The examples of pattern-to-pattern mapping (global nature of cells' states) are shown in Fig. 1.

Even though the system size n becomes infinitely large, namely, ∞, the set of points in mapping is measure zero in Lebegue sense, or has only the density of countable number at the most. We intended to enlarge an isolated point into a local line, namely, to extend measure zero mapping into measurable one. To call the map as a fractal map, we also demand self-similarity [2] of mapping.

In the next section, we show the procedure to construct the 1-d fractal map using scaling and location shift transformations. In section 3, we consider dynamics using 1-d fractal maps. More complicated orbits are generated 1-d fractalmaps, compared with the orbits by the ordinal 1-d maps.

2. Construction Procedure of 1-D Fractal Map

The essential nature of fractal is the self-similarity [2]. Rescaling and embedding of any figure realize the fractal nature. We therefore use scaling transformation and location shift transformation to construct a 1-d fractal map.

Let consider 1-d map $f(x)$ defined in the interval $\mathrm{I}=[0, 1]$. This is sufficient to realize any 1-d map in any interval by appling scaling and location shift transformations for the 1-d map defined in the interval I. The 1-d mapping is represented as $f: \mathrm{I} \to \mathrm{I}$ or $y=f(x)$, $(x, y \in \mathrm{I})$.

2.1 Scaling transformation

The scaling transformation $\mathrm{S}(\mu, \nu)$ applied for a 1-d map $f(x)$ is defined as follows:

$$\mathrm{S}(\mu, \nu)\cdot f(x) \equiv f_{\mathrm{scale}}(x) = \nu f(x/\mu) \tag{2.1}$$

where μ, ν are scaling factors for x and y directions, respectively. Schematical illustration is shown in Fig. 2. After application of scaling transformation, the mapping becomes $f_{\mathrm{scale}}: \in \mathrm{I}(\mu) \to y \in \mathrm{I}(\nu)$, $(\mathrm{I}(\mu)=[0, \mu], \mathrm{I}(\nu)=[0, \nu])$. When $\mu<1$ and $\nu<1$, the mapping is contractive.

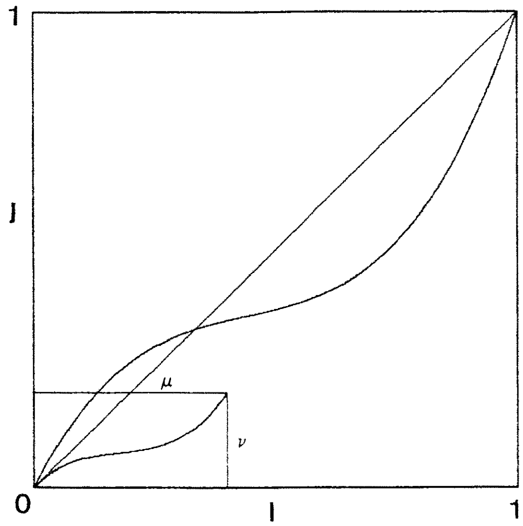

Fig. 2 An Illustration for Scaling Transformation

2.2 Location shift transformation

The location shift transformation $\mathrm{L}(a, b)$ applied for a 1-d map $f(x)$ is also defined as follows:

$$\mathrm{L}(a, b)\cdot f(x) \equiv f_{\mathrm{shift}}(x) = f(x-a)+b, \tag{2.2}$$

where a, b denote location shift coordinate (a, b) for the origin $(0, 0)$ of original function $f(x)$ defined in the interval I.

2.3 Embedding transformation

We call both application of scaling and location shift transformations for an original

1-d map $f(x)$ as the embedding transformation. Then the embedding transformation is expressed as

$$L(a, b)\cdot S(\mu, \nu)\cdot f(x) \equiv f_{LS}(x) = \nu f((x-a)/\mu) + b. \qquad (2.3)$$

The mapping embedded-transformed becomes $f_{LS}: x \in I(\mu, a) \rightarrow y \in I(\nu, b)$, $(I(\mu, a) = [a, \mu + a]$, $I(\nu, b) = [b, \nu + b])$. Notice that the application order of scaling and location shift transformations is trivial because of $L(a, b)\cdot S(\mu, \nu)\cdot f(x) = S(\mu, \nu)\cdot L(a, b)\cdot f(x)$. The illustration of embedding transformation is depicted in Fig. 3.

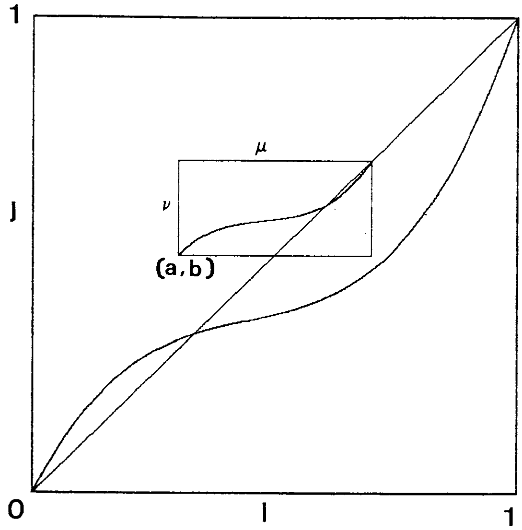

Fig. 3 An Illustration for Location Shist Transformation

2.4 The procedure to construct a 1-d map

Using embedding transformation, we can construct a 1-d fractal map. The 1-d fractal map is a kind of discontinuity maps. A discontinuity map is consisting of a set of submaps as schematically shown in Fig. 4. The number of discontinuity points in a map is less than the countable number of infinite. The 1-d discontinuity mapping M_d is formally described as follows:

$$M_d = \{g(k),\ k = 1, \cdots, m;\ m \leqq \infty\},$$
$$M_d: I \rightarrow J,\ (I = J = [0, 1]),$$
$$g(k): I(k) \rightarrow J(l),\ (I(k) \in I = \Sigma I(k),\ J(l) \in J = \Sigma J(l),\ I(k) \cap I(k') = \phi,\ J(l) \cap J(l') = \phi), \qquad (2.4)$$

where $g(k)$ denotes submaps, $I(k)$ and $J(l)$ are sub-intervals of I and J, respectively.

To make the discontinuity map in the manner where submaps are arranged self- similar feature, we can obtain a fractal map. There exist many methods to bring the fractal nature

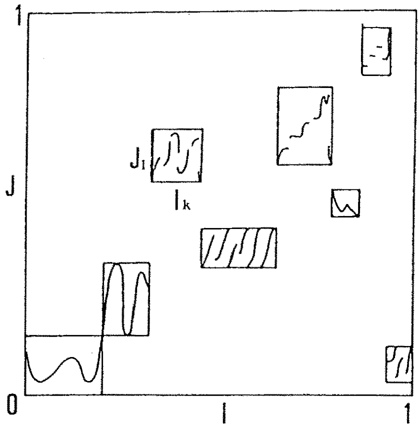

Fig. 4 An Illustration for Discontinuity Map

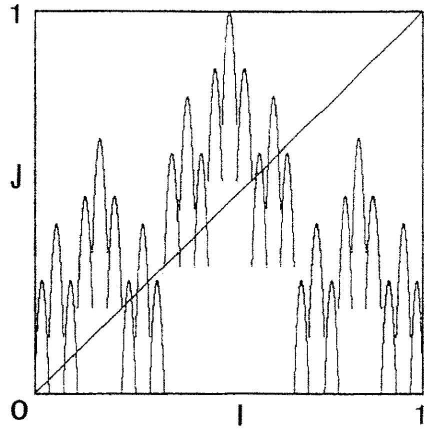

Fig. 5 An Example of One-Dimensional Fractal Map
The original for embedding transformation is logistic map.

(namely, a self-similarity) on a discontinuity map. One of the methods to make a self-similar map is repetitious application of embedding transformation. Another is to place the contracted maps side by side in a whole 1-d fractal map by self-similar manner. An example is shown in Fig. 5 using the contracted logistic map.

3. Iterative Dynamics on A 1-D Fractal Map

Let an 1-d fractal map be denoted by $M_f(x)$. The dynamics for a 1-d fractal map is the generation of orbits by iterative application of $M_f(x)$. Then the equation to describe the dynamics has following recurrent form

$$x_{n+1}=M_f(x_n) \tag{3.1}$$

where n means iteration number and initial value x_0 be assumed for this equation.

In order to consider an orbit generated by the $M_f(x)$, it is useful to classify the submaps into two classes. The classes of submaps are state region and transition region. The classification criterion is very simple, namely,

State region: the submap which crosses the line $x_{n+1}=x_n$ or tangential structure for the same line at any point is classified into this class,

Transition region: the submap which has no crossing point or no tangential structure for the line $x_{n+1}=x_n$ is classified into this class.

When the orbit entered into the transition region, it immediately goes out the submap classified into transition region at one iteration. On the other hand, the orbit stays at least a few times in the state region. If the submap in the state region has the tangential structure, the orbit stays for a long time in the state region. We can see that the complicated orbit is more easily appeared for the 1-d fractal map.

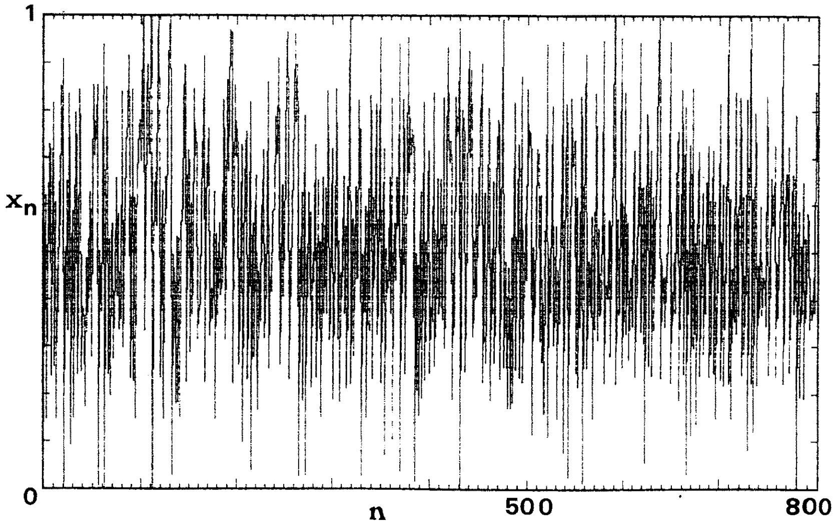

Fig. 6 An Example of Orbit Generated by the 1-D Fractal Logistic Map.

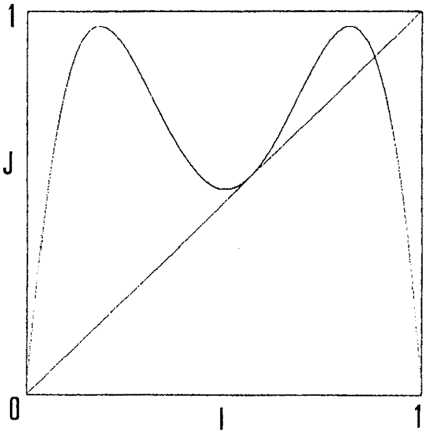

Fig. 7 One-Dimensional M Map

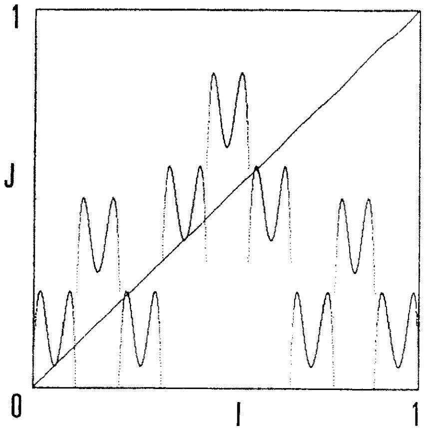

Fig. 8 One-Dimensional Fractal M Map
This map is given by two times repetition of the 1-d M map shown in Fig. 6

Figure 6 shows an orbit generated by the fractal logistic map (Fig. 5). To make the tangential structure in the map, we introduce the M map shown in Fig. 7. More complicated orbits generated by the fractal M map (Fig. 8) are shown in Figs. 9-11. A complicated intermittent orbit is appeared in the fractal M map (Fig. 11). This implies that chaotic itineracy of an orbit is latenet in the dynamics for the 1-d fractal maps.

4. Discussion

As stated in the last of previous section, the chaotic itineracy should be appeared in the

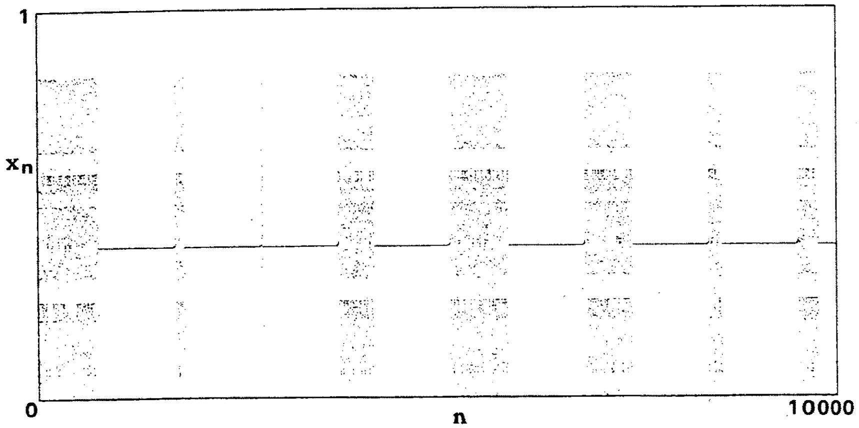

Fig. 9 An Intermittent Orbit for the 1-D Fractal M Map
Chaotic and steady local orbits are alternated in chaotic manner.

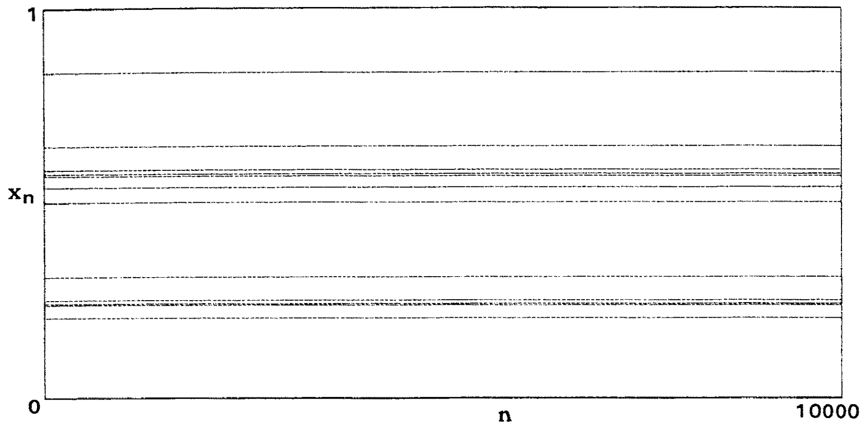

Fig. 10 A Multi-Periodic Orbit for the 1-D Fractal M Map

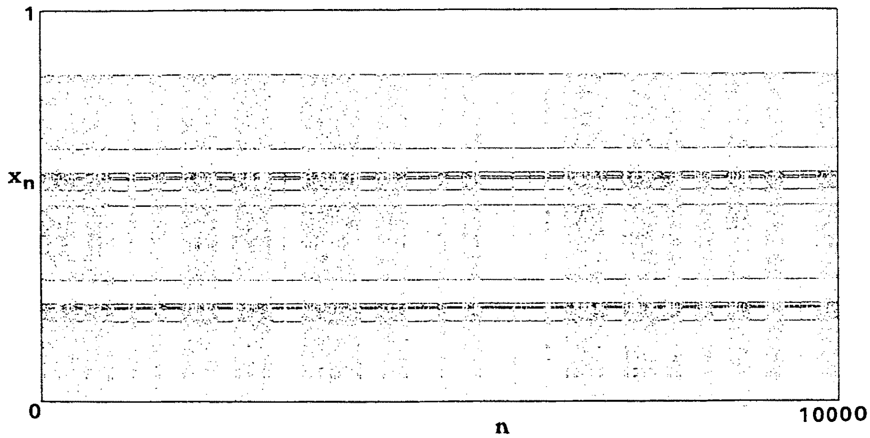

Fig. 11 A Complicated Intermittent Orbit for the 1-D Fractal M Map
Chaotic and multi-periodic local orbits are alternated in chaotic manner.

1-d fractal maps. It is also expected that the orbit which goes its rounds of several state regions in the manner of chaotic itineracy may appear. This nature may also be realized in other class of discontinuity maps. The self-similarity is not needed. It is essential that the map is everywhere discontinuous. The requirement of self-similarity for the 1-d fractal map may bring the more simple equations to realize the chaotic itineracy of orbits.

Now we consider the physical reality of 1-d fractal map. We expect that the 1-d fractal map is obtainable when we make a return map from the Poincaré section of reconstructed attractor embedded by a vector time series data [3,4]. If the mechanics of motion has muti-components, the manifold constructed by orbits may be multi-fractal structure. When the orbits go their rounds sub-components fields, the manifold becomes complicated structure of a Cantor set. We therefore may obtain multi-fractal structure of Poincaré section. This fact may leads to the fractal return map on a Poincaré section.

References

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