Paper

Deterministic Chaos in Japanese Economic Variables

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Abstract: In this paper we show some evidences which indicate time series data of some Japanese economic variables behave like deterministic chaos at least for some period, using a return map of intersecting points on a Poincaré section, i.e., a plane which cuts an embedded manifold constructed by these variables. We propose possible dynamic structures for the economic variables indicating the nature of deterministic chaos and discuss them.

1. Introduction

We have been studying time series data of economic variables from a viewpoint of deterministic chaos[1–5]. We began by studying correlation dimensions for actual economic data in Japan. Since size of the economic data is very small, we examined at first whether or not the Grassberger–Procaccia's method (G–P method)[6] is applicable to such small sized data[7]. The applicability of the G–P method to small sized data had already been examined by other researchers[8,9]. We calculated correlation integrals to obtain correlation dimensions and found that some economic variables had finite values of the correlation dimension[3]. Grassberger, however, showed that small sized data of random number sequence also had a finite value of correlation dimension[10]. Accordingly, he suggested that a finite value of correlation dimension did not always imply deterministic chaos[10]. It means, in effect, it is difficult in general to judge deterministic chaos only from the fact that time series data have a finite value of correlation dimension calculated by the G–P method.

By this reason, we must have looked for an alternative method to judge deterministic chaos of actual data. The method presented in this paper is based on a primitive idea. It is associated with the attractor of a Lorenz model[11] constructed by embedding three-dimensional sequential data in three-dimensional space. We have attempted to embed two or three sets of time series data into two- or three-dimensional space, respectively[4,5]. The determinisity was examined by analyzing structures of a Poincaré section of the embedded attractor. If the attractor is constructed by a chaotic orbit, a return map of intersecting points of the orbit on the Poincare section must have a simple structure like one-dimensional mapping[5]. In an actual fact, we obtained 1-d like return maps of the intersecting points on the

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Poincare sections for some economic variables, e.g., GDP, PHI (private housing investment), PEI (private equipment investment), as shown below. In this paper we will present these return maps to show that some Japanese economic variables may have a nature of deterministic chaos.

In the next section, we will briefly explain the method used in this study and then show some examples of embedded attractors, distribution of intersecting points on a Poincaré section, and return maps of the intersecting points.

In section 3, we will present return maps which indicate deterministic chaos, together with embedded attractor and intersecting points on a Poincaré section, for Japanese economic data. Return maps which appear 1-d like mapping are classified into two classes by their distinctive features.

In section 4, we will discuss deterministic nature of economic variables conjectured by examining the return maps of intersecting points on a Poincaré section. We also discuss how we should interpret ambivalence occurring in the classification of the return maps defined in section 3.

2. Embedded Attractor of Detrended Japanese Economic Variables.

Time delay coordinate embedding is usually used to reconstruct an attractor from one-dimensional sequential data[12,13]. Here we present another embedding procedure to construct a manifold from three sets of time series data[4,5]. The embedded manifold is obtained by plotting a point (x_i, y_i, z_i) for three economic variables at a given time in three-dimensional space and by connecting them sequentially. Although the idea is very simple, much information can be derived from this embedding method.

The data of Japanese economic variables used in this study were obtained from "Annual Reports on National Accounts" by the Economic Agency, Government of Japan. The economic variables analyzed in this study are GDP, PFCE (private final consumption expenditure), PEI (private equipment investment), PHI (private housing investment), EXP (export), and IMP (import). The period used for the analyses is from the first quarter of 1955 to the first quarter of 1993. The total number of data points is 153.

The original data are trended ones. Since the trended data are not suitable for our analysis to study chaos dynamics, a growth rate r_t, instead of the original (i. e., trended) data Q_t, is used in this study. r_t is defined as

$$r_{t+1}=\frac{Q_{t+1}-Q_t}{Q_t}=\frac{Q_{t+1}}{Q_t}-1. \tag{2.1}$$

Notice that growth rate r_t takes both positive or negative value, even if we call r_t growth rate. The data expressed by the growth rate of economic variable is a kind of detrended ones. The original (trended) data and growth rates of GDP, PFCE, PEI, PHI, EXP, and IMP are shown in **Figs. 1.1–1.3**.

We construct an embedded manifold by picking up three out of the six economic varia-

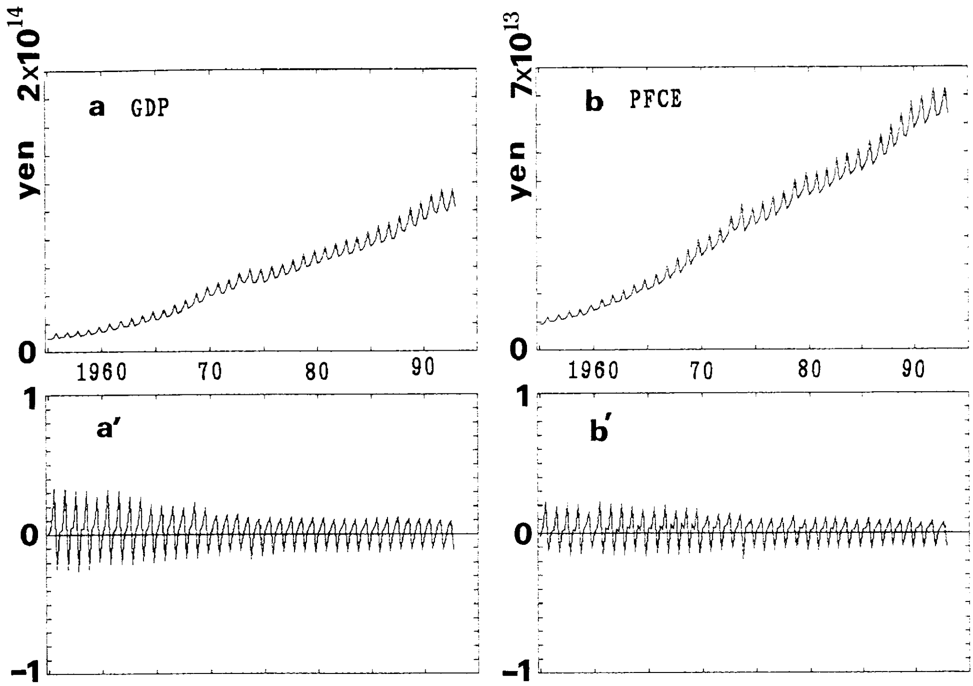

Fig. 1.1 Time Series of Quarterly Trended Data and Their Growth Rates. (a) GDP, (b) PFCE; (a′), (b′) growth rate of (a), (b), respectively. The data for the period from the first quarter of 1955 to the first quarter of 1993 are used for the analysis.

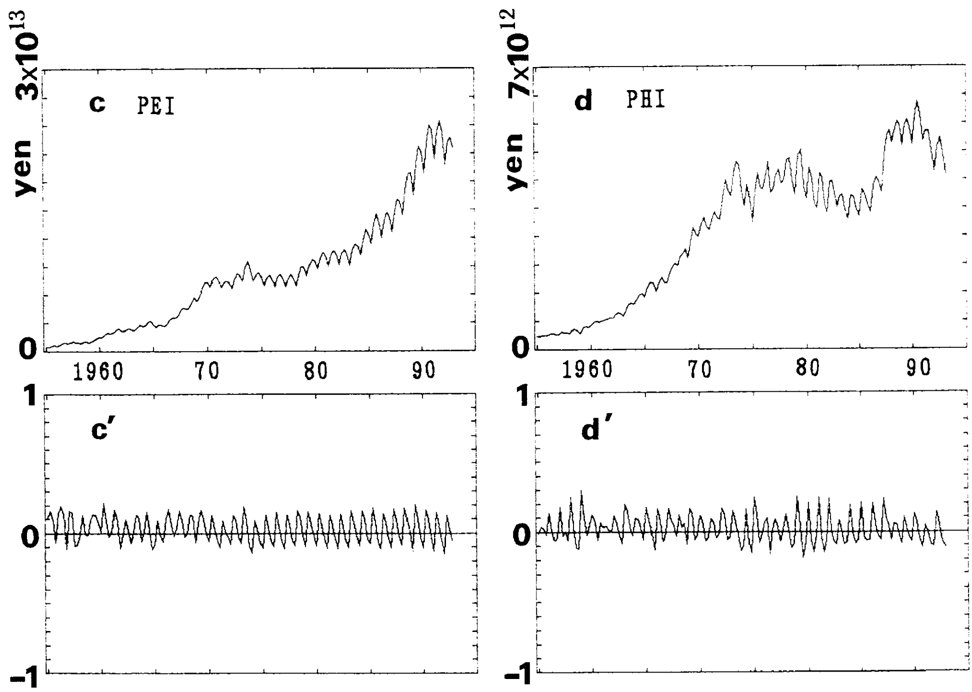

Fig. 1.2 The same as Fig. 1.1, but for (c) PEI, (d) PHI; (c′), (d′) growth rate of (c), (d), respectively.

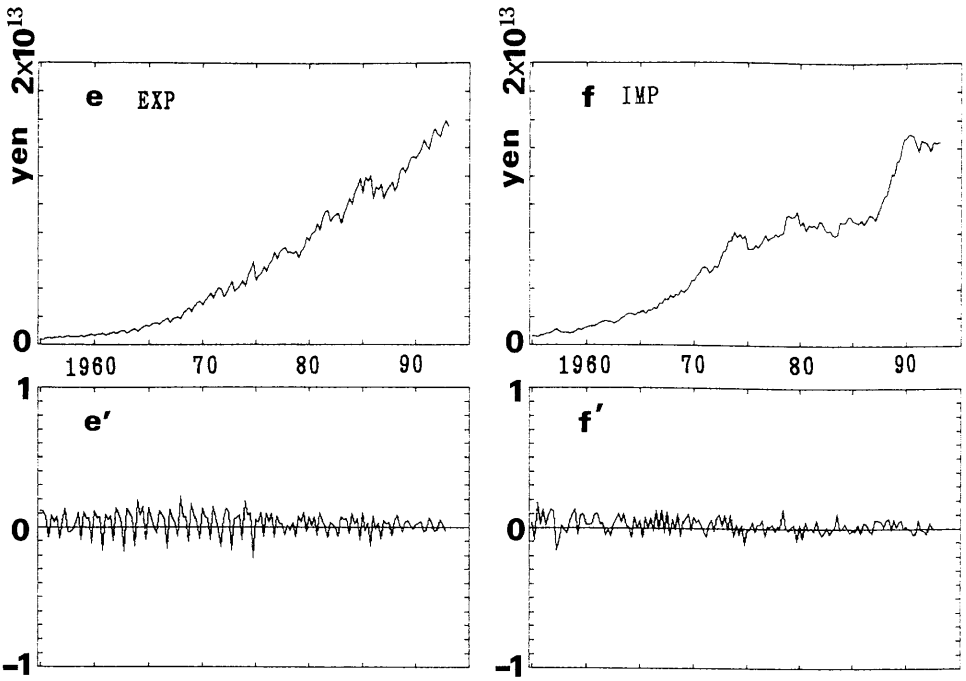

Fig. 1.3 The same as Fig. 1.1, but for (e) EXP, (f) IMP; (e′), (f′) growth rate of (e), (f), respectively.

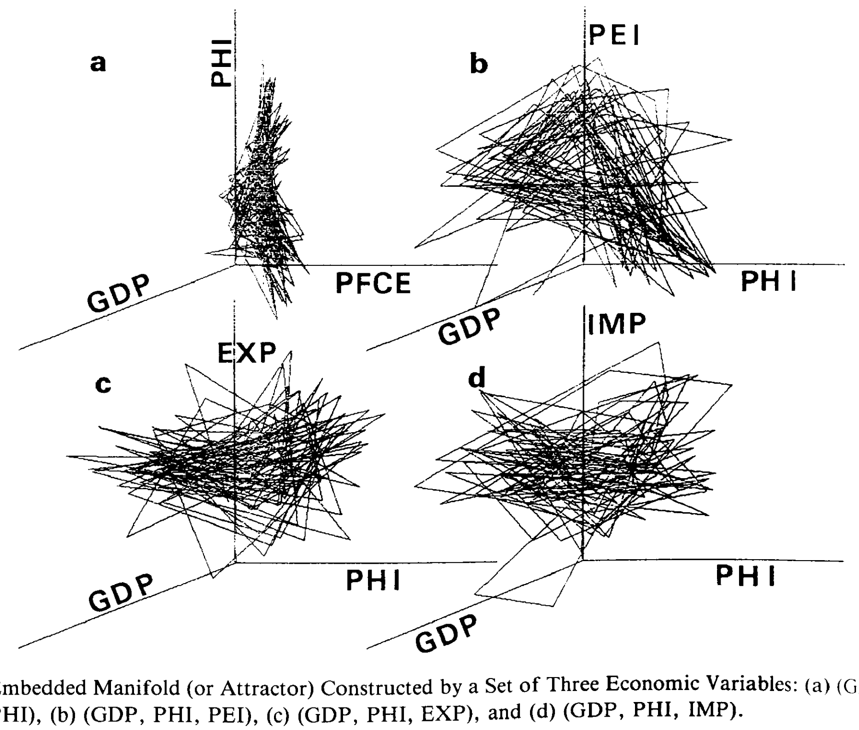

Fig. 2 Embedded Manifold (or Attractor) Constructed by a Set of Three Economic Variables: (a) (GDP, PFCE, PHI), (b) (GDP, PHI, PEI), (c) (GDP, PHI, EXP), and (d) (GDP, PHI, IMP).

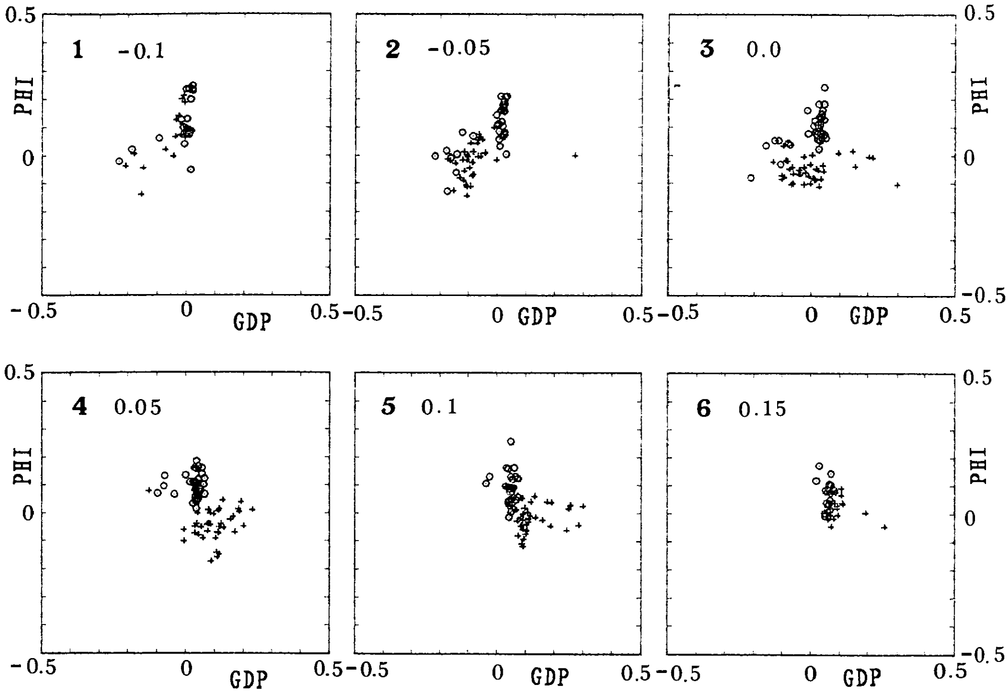

Fig. 3 Distribution of Intersecting Points on a Poincare Section (GDP, PHI) for Embedded Manifold (GDP, PHI, PEI). PEI is set to (1) −0.1, (2) −0.05, (3) 0.0, (4) 0.05, (5) 0.1, and (6) 0.15.

bles. The number of ways to select three out of six is ${}_6C_3=20$. **Fig. 2** shows four examples of the embedded manifold (or embedded attractor) of the growth rate for the sets of economic variables, (GDP, PFCE, PHI), (GDP, PHI, PEI), (GDP, PHI, EXP), and (GDP, PHI, IMP).

To examine the structure of the embedded manifold, it is useful to cut it by a plane. For example, we cut the embedded manifold in an orthogonal coordinate system of three variables x, y, and z by a plane parallel to x–y, y–z, or z–x plane. We are interested in the points where an orbit intersect the plane. The plane with these points (referred to as intersecting points) is called Poincaré section in this paper. Actual procedure to generate intersecting points is shown in refs.[4,5]. Examples are shown in **Fig. 3**, for a set of three economic variables (GDP, PHI, PEI) in Japan (see also Fig. 2b for this set of variables).

In order to explore deterministic nature of an orbit constructing an embedded manifold, the return map of intersecting points is useful. Let coordinates of a plane cutting the embedded manifold be (ξ,η). Intersecting points are obtained as sequential data $\{(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_i, \eta_i), \ldots\}$. We therefore make two return maps of the intersecting points, i.e., (ξ_i, ξ_{i+1}) and (η_i, η_{i+1}). An interval of intersecting time $(\tau_{i+1}-\tau_i)$ is also interesting; intersecting time is defined as the time at which the orbit in the embedded manifold intersects the cutting plane. A return map of the intersecting time intervals is also drawn to study the na-

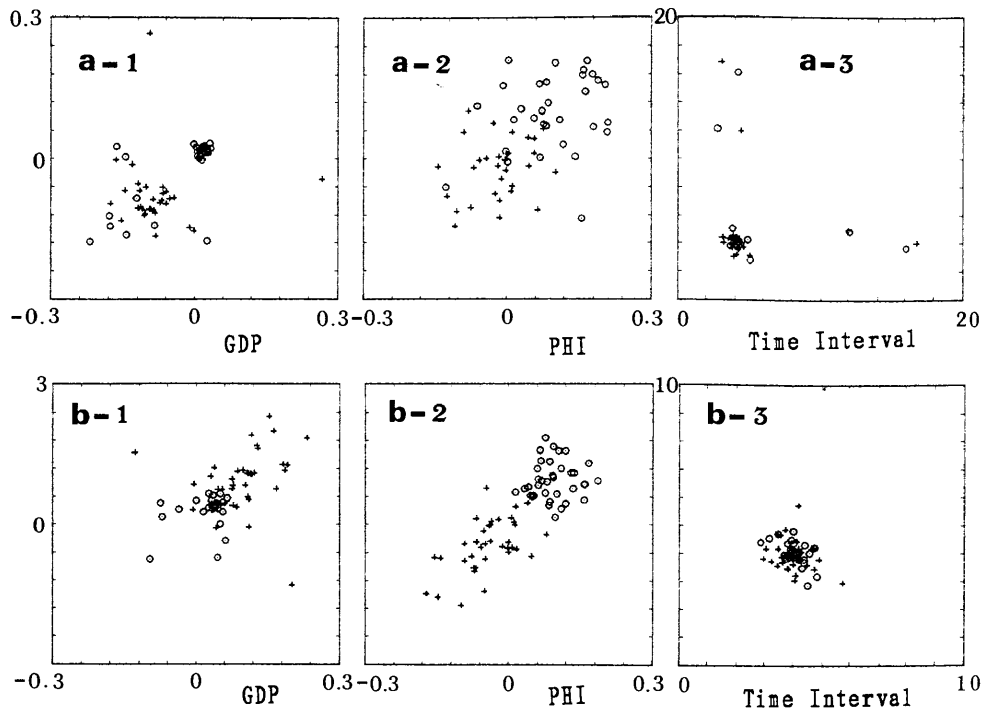

Fig. 4 Return Maps of Intersecting Points and Intersecting Time Intervals on a Poincaré Section (GDP, PHI) for Embedded Manifold (GDP, PHI, PEI). PEI is set to (a) −0.05 and (b) 0.05. (1), (2), and (3) are return maps of GDP, PHI, and intersecting time intervals, respectively.

ture of the orbit. Two examples for these return maps are shown in **Fig. 4**.

Investigations of deterministic chaos suggest that one-dimensional feature should be observed in the return map, if the irregular behavior of a given system is governed by a certain deterministic law[11,14]. In other words, there exists some one- dimensional mapping, i.e., $\xi_{i+1}=f(\xi_i)$, $\eta_{i+1}=g(\eta_i)$, for the intersecting points on the Poincaré section. In this context, we use the following criterion for judgment of deterministic chaos; *a set of time series data is regarded to have a nature of deterministic chaos, if a return map reveals one-dimensional feature.*

3. Deterministic Nature of Japanese Economic Variables.

In this section we show some evidences of deterministic chaos in some Japanese economic variables. As mentioned in section 2, a set of three economic variables constructs an embedding manifold, and the return maps of intersecting points and intersecting time intervals on the plane cutting the manifold may provide information about determinisity in the data. The results obtained by analyzing the actual Japanese economic data with the above method, are shown in **Figs. 5–16**. Each figure contains embedded manifold (upper left), dis-

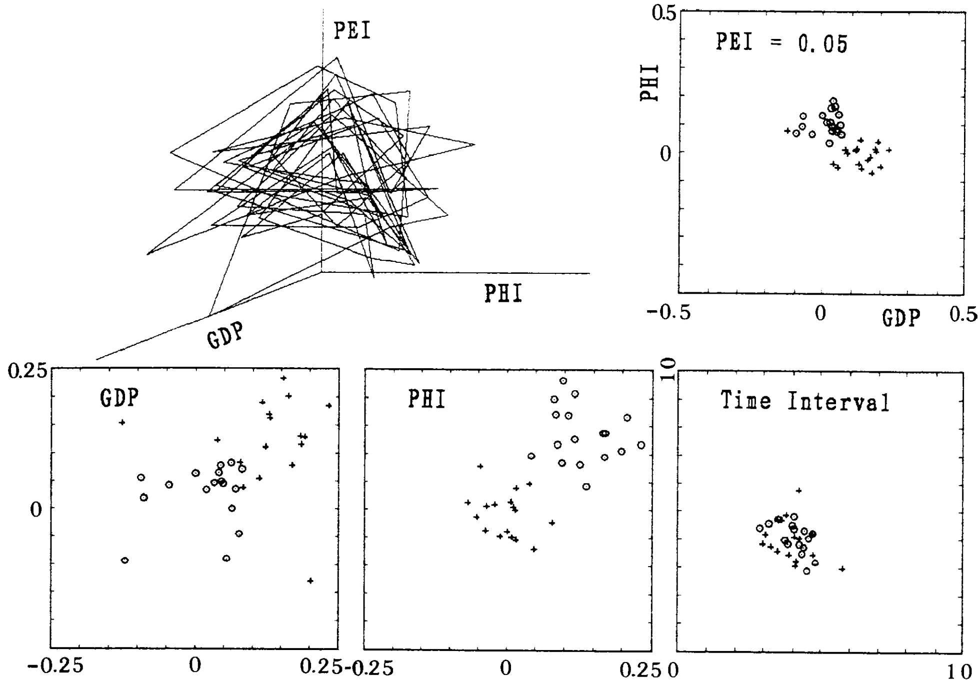

Fig. 5 Poincaré Section (Upper Right), Return Maps of GDP, PHI, and Intersecting Time Intervals (Lower Three Maps) for Embedded Manifold (GDP, PHI, PEI) of Period I. PEI is set to 0.05.

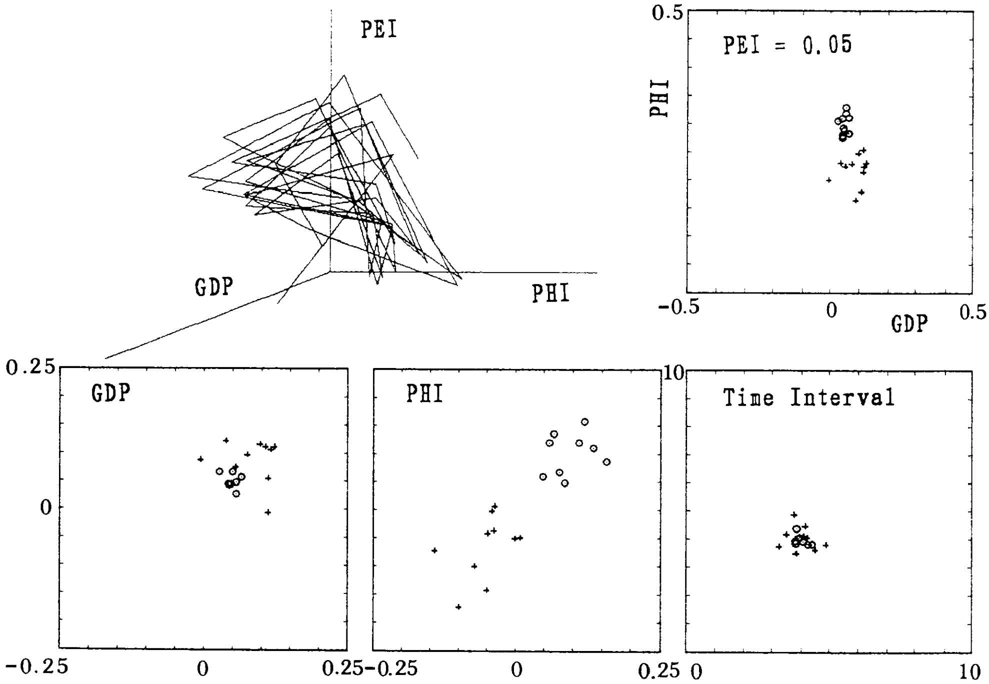

Fig. 6 The Same as Fig. 5, but for Period II.

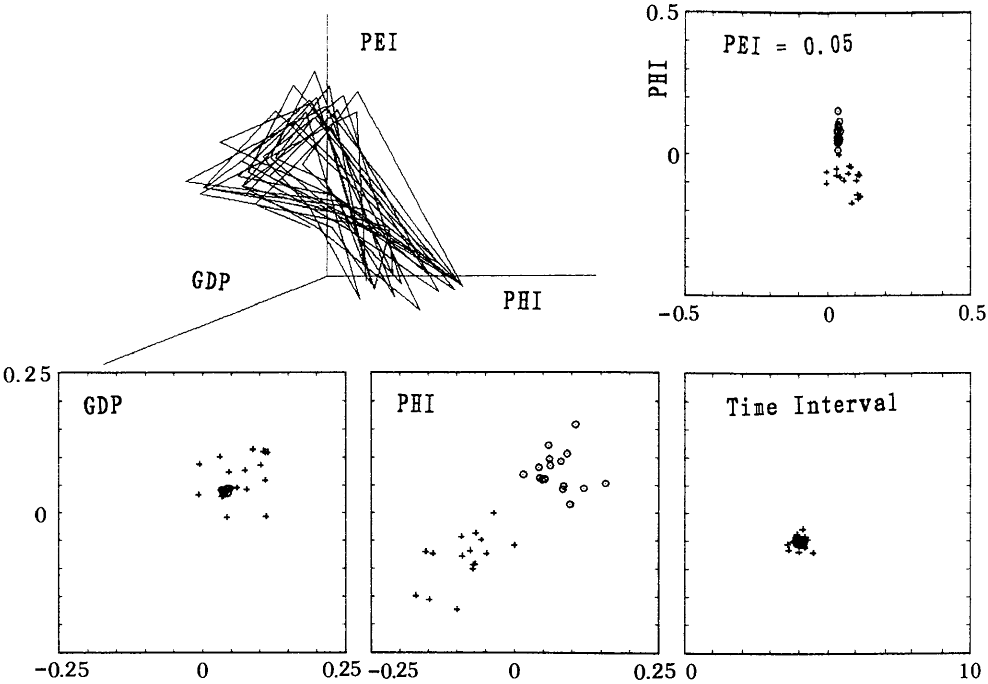

Fig. 7 The Same as Fig. 5 but for Period III.

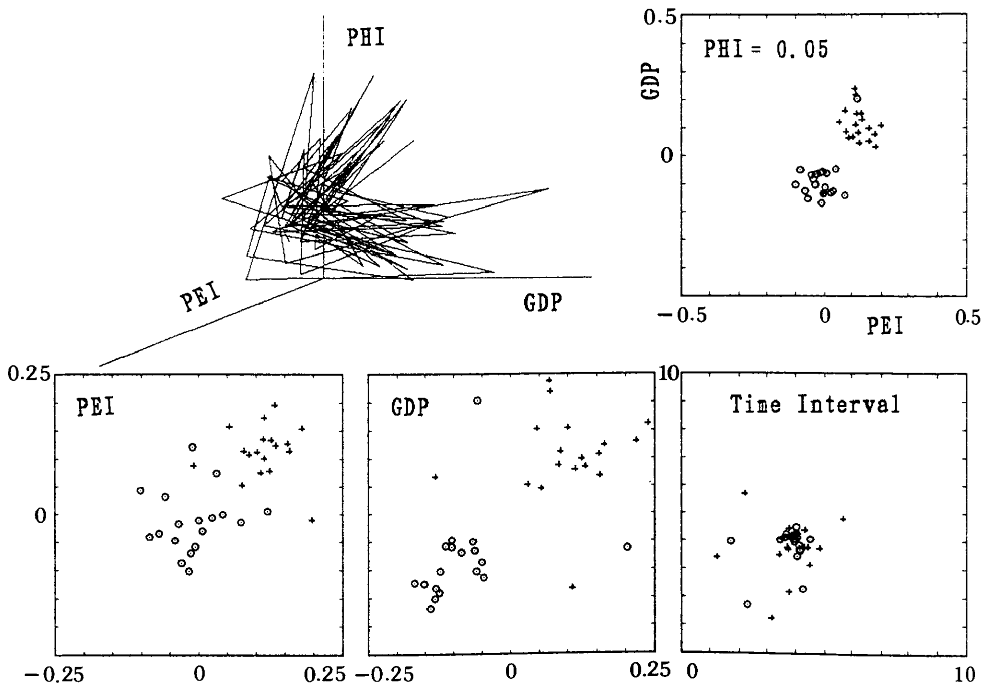

Fig. 8 Poincaré Section (Upper Right), Return Maps of PEI, GDP, and Intersecting Time Intervals (Lower Three Maps) for Embedded Manifold (PEI, GDP, PHI) of Period I. PHI is set to 0.05.

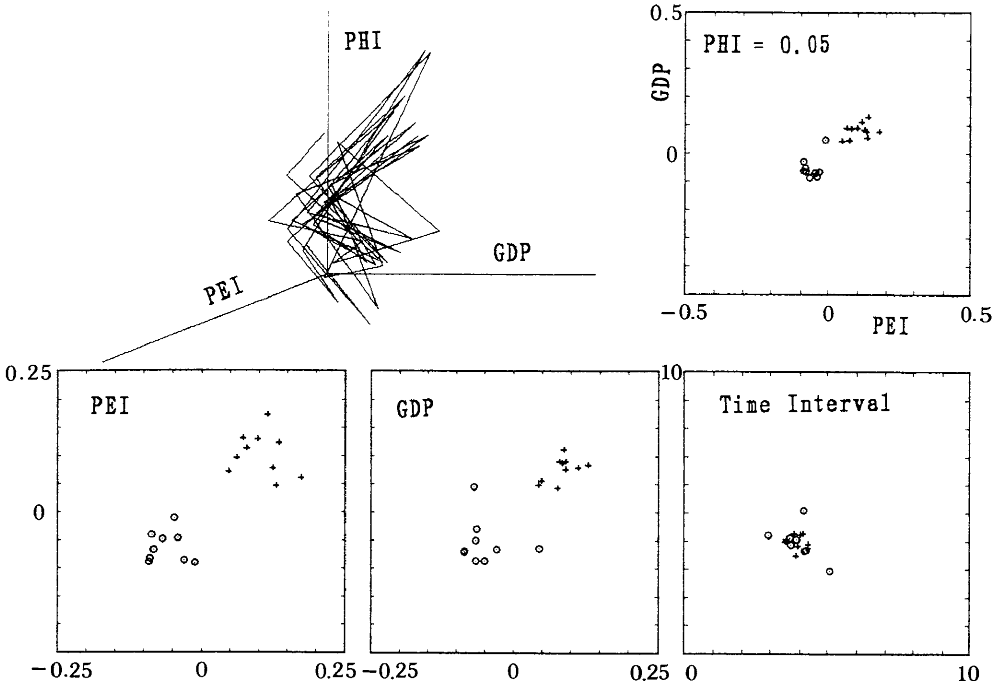

Fig. 9 The Same as Fig. 8, but for Period II.

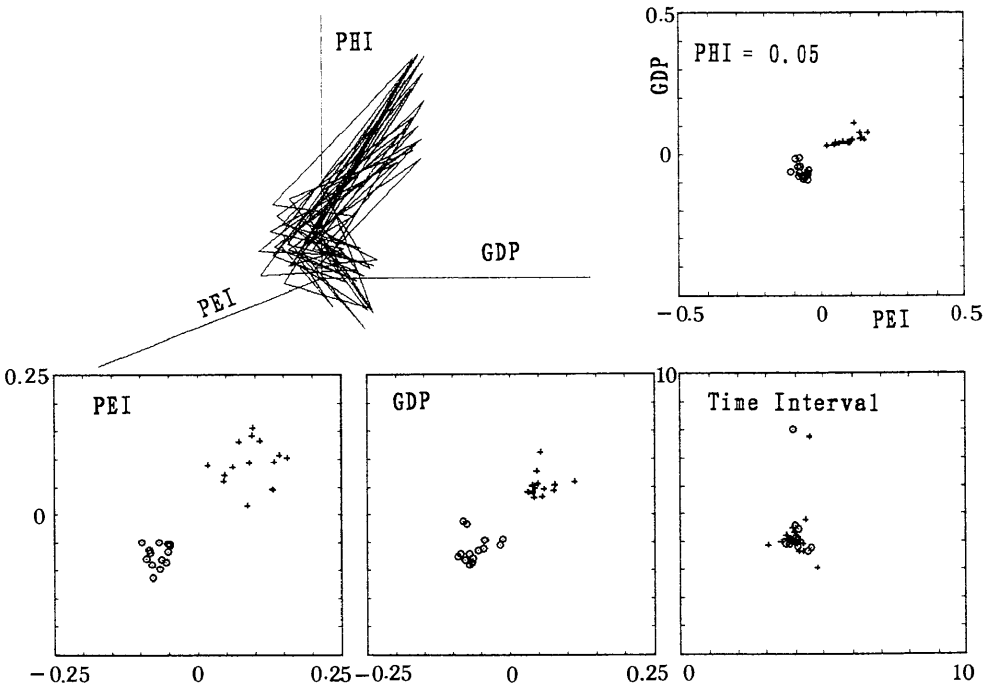

Fig. 10 The Same as Fig. 8, but for Period III.

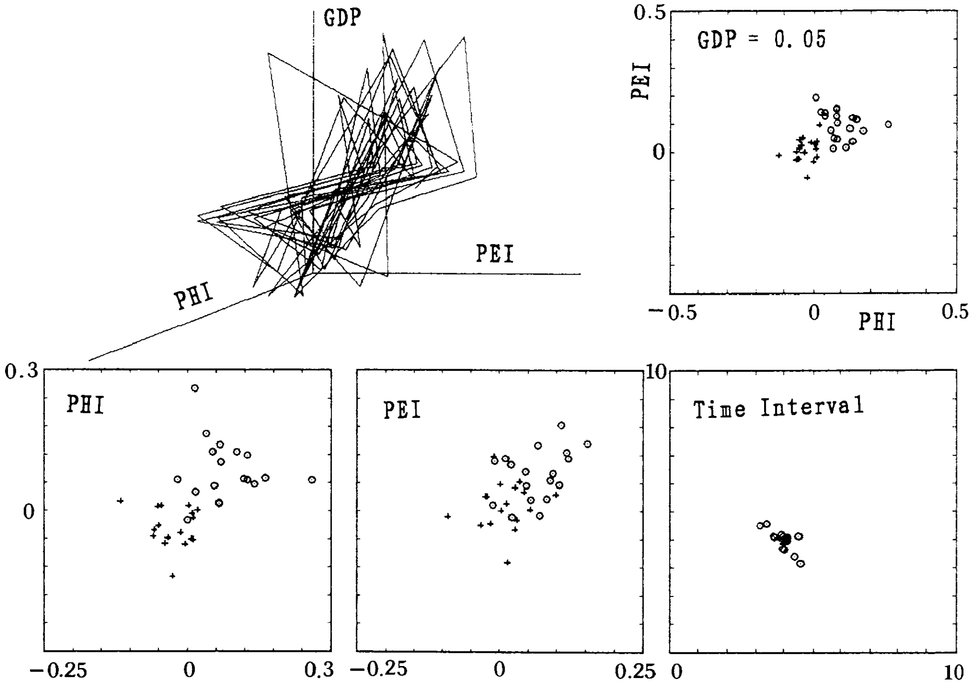

Fig. 11 Poincaré Section (Upper Right), Return Maps of PHI, PEI, and Intersecting Time Intervals (Lower Three Maps) for Embedded Manifold (PHI, PEI, GDP) of Period I. GDP is set to 0.05.

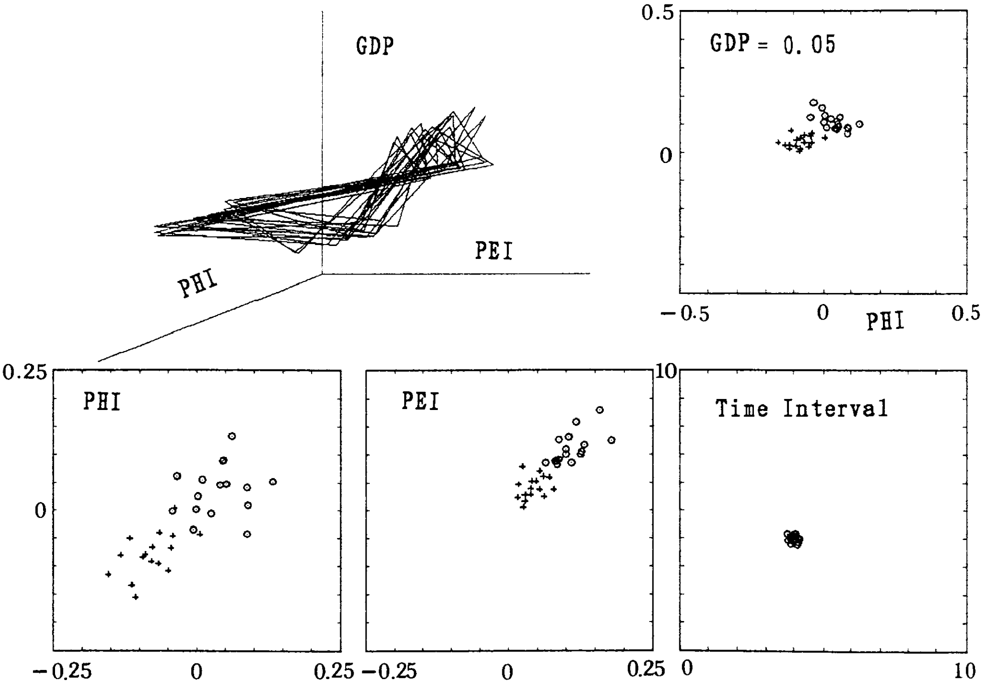

Fig. 12 The Same as Fig. 11, but for Period II.

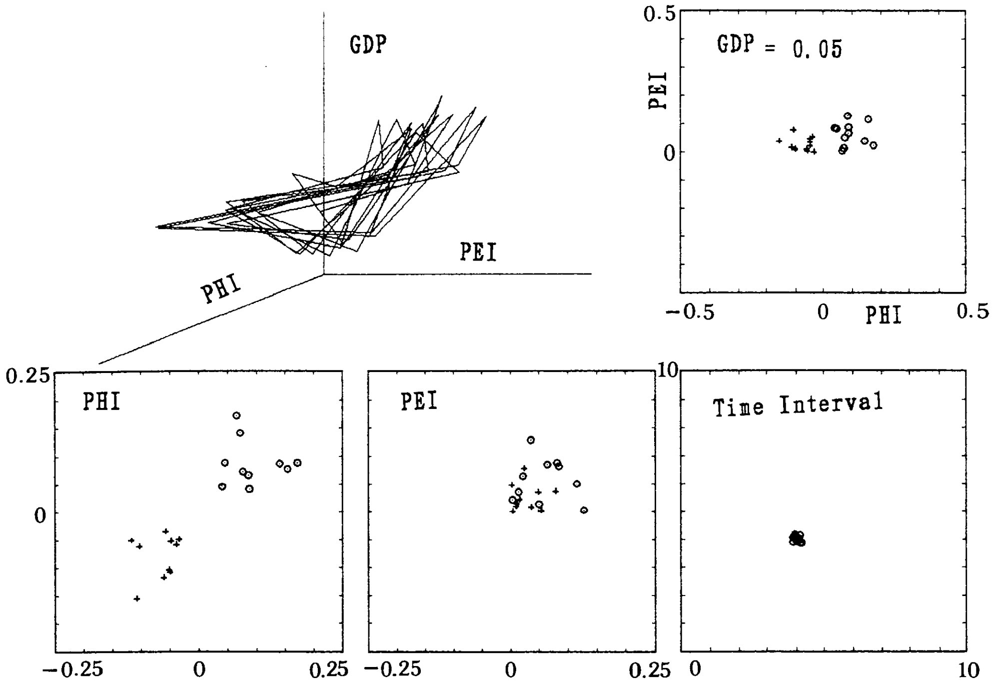

Fig. 13 The Same as Fig. 11, but for Period III.

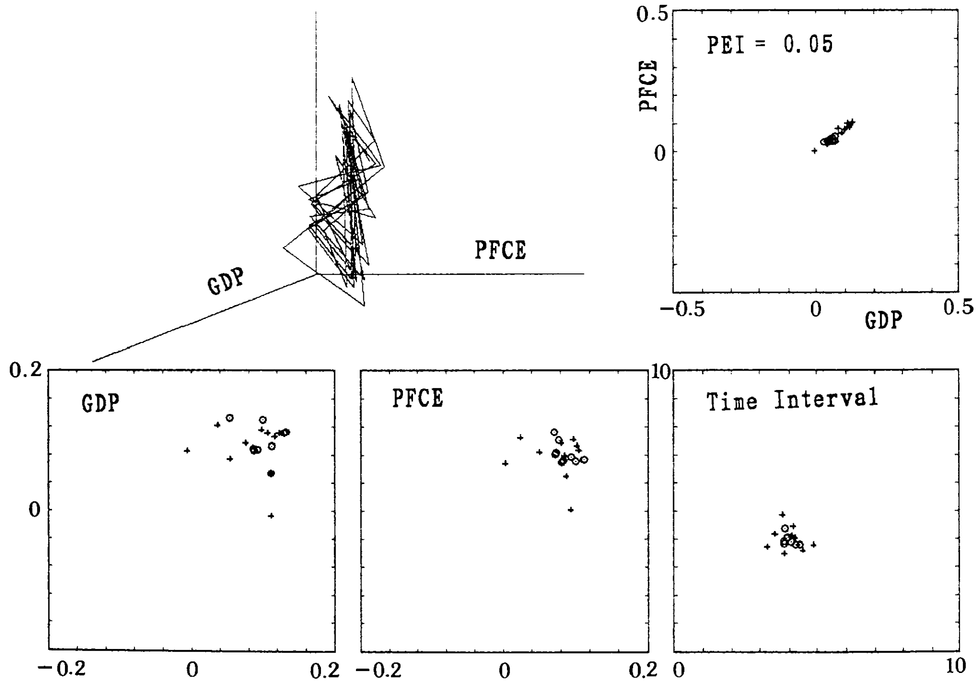

Fig. 14 Poincaré Section (Upper Right), Return Maps of GDP, PFCE and Intersecting Time Intervals (Lower Three Maps) for Embedded Manifold (GDP, PFCE, PEI) of Period II. PEI is set to 0.05.

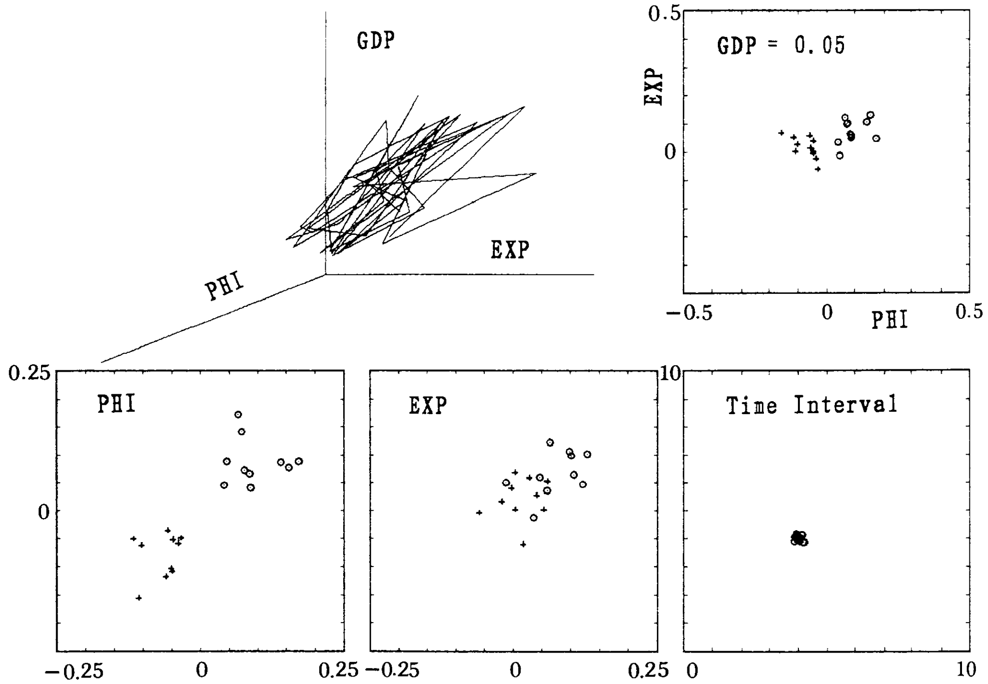

Fig. 15 Poincaré Section (Upper Right), Return Maps of PHI, EXP and Intersecting Time Intervals (Lower Three Maps) for Embedded Manifold (PHI, EXP, GDP) of Period II. GDP is set to 0.05.

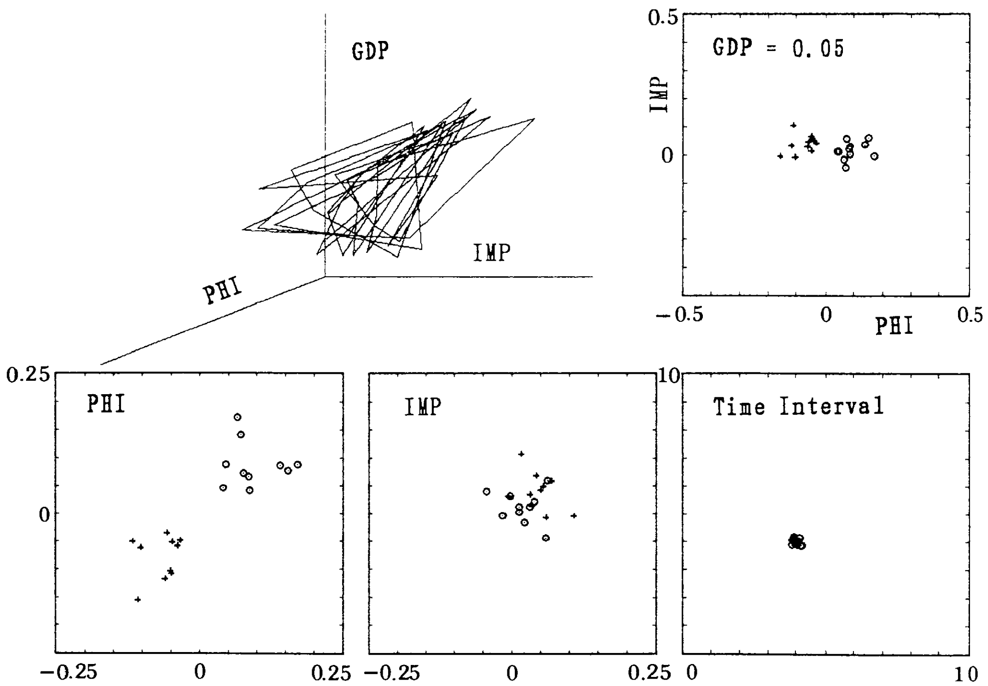

Fig. 16 Poincaré Section (Upper Right), Return Maps of PHI, IMP and Intersecting Time Intervals (Lower Three Maps) for Embedded Manifold (PHI, IMP, GDP) of Period II. GDP is set to 0.05.

tribution of intersecting points (upper right), two return maps corresponding to intersecting points (lower left and lower center), and a return map of intersecting time intervals (lower right).

From an economic point of view, it is interesting to divide economic data into three periods, namely, period I: before the first oil shock (from the second quarter of 1955 to the third quarter of 1973), period II: period including the first oil shock (the first quarter of 1969 to the forth quarter of 1979), and period III: after the first oil shock (from the first quarter of 1976 to the first quarter of 1993). The first and second periods, and the second and third periods have overlapped periods, respectively. We expected that the analysis by dividing the date into the three periods might provide some aspects of deterministic nature of the time series data of the economic variables, because our preliminary analysis of (GNP, PFCE, PEI)[4] and (GNP, PFCE, PHI)[5] indicated that the return map of PFCE in period II (in which the first oil shock occurred) was like 1-d mapping.

In this study we examine only a Poincaré section parallel to $x-y$, $y-z$, or $z-x$ plane for an embedded manifold constructed by three economic variables, x, y and z. As an example, the analyses of the embedded manifold of (GDP, PHI, PEI) are shown in **Figs. 5–13**. In Figs. 5, 6, and 7, the return maps of GDP and PHI on the Poincaré section at PEI$=$0.05 for periods I, II, and III are shown, respectively. In Figs. 8, 9, and 10, the return maps of PEI and GDP on the Poincaré section at PHI$=$0.05, and in Figs. 11, 12, and 13, the return maps of PHI and PEI on the Poincaré section at GDP$=$0.05 are shown for periods I, II, and III, respectively.

Deterministic feature is found for the return maps of GDP for period I (Fig. 5), PHI for period II (Fig. 6), GDP for period I (Fig. 8), PEI and GDP for period II (Fig. 9), and PHI for period II (Fig. 12). In these figures the intersecting points are shown by the two kind of symbols, circle and cross, depending on the directions in which the orbit intersects the Poincare section. The deterministic feature mentioned is restricted to the intersection points in upward from below the plane (denoted by circle). The intersecting points in the downward direction from above the plane (denoted by the cross symbol) reveal deterministic feature in some return maps, e.g., those of GDP for period III (Fig. 10) and PEI for period II (Fig. 9).

The same analyses were carried out for other triplets of the economic variables shown in Fig. 2. The results of the analyses for (GDP, PFCE, PEI), (GDP, PHI, EXP), and (GDP, PHI, IMP) for period II are shown in **Figs. 14, 15**, and **16**, respectively. The return maps of PHI in Figs. 15 and 16, on the Poincaré section at GDP$=$0.05, imply some determinisity of the data.

In the result of examining the return maps obtained in this way, we consider that the return maps of Japanese economic variables which are regarded to have a chaotic nature can be classified into two groups. One group has a feature of the return map shown in **Fig. 17a**, and the other has a unimodal feature of the return map shown in **Fig. 17b**. Since the unimodal return map shown in Fig. 17b generates unbounded time series data (except for a

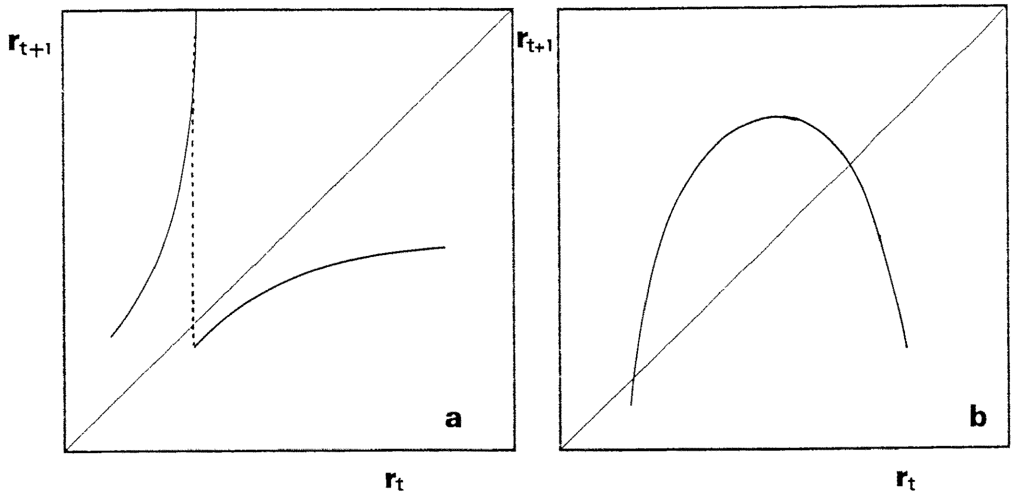

Fig. 17 Schematic Drawings for Two Kinds of Return Maps. (a) Class I: idealizing the results of GDP for period I (Fig. 5), GDP for period II (Fig. 9), and PHI for period II (Fig. 12).
(b) Class II: idealizing the results of PHI for period II (Fig. 6), GDP for period I (Fig. 8), and PEI for period II (Fig. 9). The return map of class II generate unbounded time series.

special case like a logistic map[12]), however, an actual feature of the return map classified into the latter group should be different from Fig. 17b. In other words, for the members classified into the latter group, we may observe only a part of the whole return map. We guess that the whole return map may have either feature of the two maps shown in **Fig. 18**. If the actual feature of the return map is the one shown in Fig. 18a, the above classification has no sense.

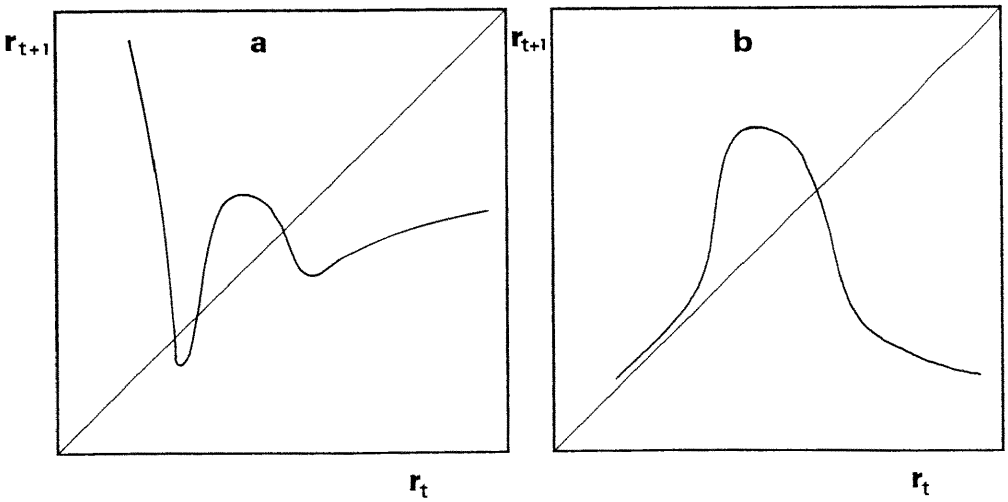

Fig. 18 Possible Scheme of Return Maps of a Bounded Type for Fig. 17b.

Even though the classification of return maps is not definite at present, we classify the return maps which reveal deterministic nature above as follows:

(Class I) Return map which has the feature like Fig. 17a.
GDP for period I in Fig. 5 (PEI=0.05),
GDP for period II in Fig. 9 (PHI=0.05),
PHI for period II in Fig.12 (GDP=0.05).

(Class II) Return map which has the feature like Fig. 17b.
PHI for period II in Fig. 6 (PEI=0.05),
GDP for period I in Fig. 8 (PHI=0.05),
PEI for period II in Fig. 9 (PHI=0.05).

It is not straightforward to derive the meaning of the above classification of the return maps. Some economic variables, e.g., GDP and PHI, are ambivalently classified. This point, i.e., reconciliation of the ambivalent nature of the economic variables in the above classification will be discussed in the next section.

4. Discussion

In the previous section it is shown that some Japanese economic variables, e.g., GDP, PHI, and PEI, may behave in a chaotic manner in some periods and that they can be classified into the two classes according to the features of the return maps. However, some variables are classified into both classes. To consider this ambivalence in the classification, we express the method in a mathematical formulation below.

Three sets of economic data are embedded into three-dimensional space to form a manifold in this study. However, the actual data are given for discrete time;

$$\{(r_x(t_i), r_y(t_i), r_z(t_i)); t_i=i\Delta T, i=0, 1, 2, 3, \cdots, n\}. \tag{4.1}$$

In order to describe an orbit with continuous time, we have to interpolate the values between the neighboring points $(r_x(t_i), r_y(t_i), r_z(t_i))$ and $(r_x(t_{i+1}), r_y(t_{i+1}), r_z(t_{i+1}))$. We simply connected the neighboring points with a straight line in this study. Then, the embedded manifold can be written as

$$\{(r_x(t), r_y(t), r_z(t)), t\in R^1\}. \tag{4.2}$$

This embedded manifold can be written in an implicit function form

$$M(r_x, r_y, r_z)=0. \tag{4.3}$$

M gives a dense subset which describes the embedded manifold (4.2). Consider the intersecting points on a cutting plane at $r_z=K$. Intersecting points (ξ, η) can be expressed in the following scheme;

$$M(\xi, \eta, K)=0 \implies M_K(\xi, \eta)=0. \tag{4.4}$$

The condition for the intersecting points (4.4) can be written in the expression like (4.2);

$$\{(r_x(\tau_k),\ r_y(\tau_k),\ r_z(\tau_k)),\ \tau_k \in \{\tau_k\} \subset \{t,\ r_z(t)=K\}\}. \tag{4.5}$$

Then, the return maps for x and y variables are expressed as follows, respectively;

$$R_K(r_x(\tau_k),\ r_x(\tau_{k+1}),\ K)=0, \tag{4.6}$$

and

$$R'_K(r_y(\tau_k),\ r_y(\tau_{k+1}),\ K)=0. \tag{4.7}$$

If the return map has one-dimensional feature, eqs.(4.6) and (4.7) can be expressed in the form

$$r_x(\tau_{k+1})=f_K(r_x(\tau_k)), \tag{4.8}$$

and

$$r_y(\tau_{k+1})=g_K(r_y(\tau_k)), \tag{4.9}$$

respectively.

A point to be considered for the problem about the ambivalence in the classification exists in the scheme(4.4). To make the consideration point more clear, we introduce a concept, flow of orbits. The flow is defined as a set of orbits close to each other near the cutting plane. We imagine a bundle of orbits intersecting the plane locally. There are many bundles of orbits, i.e., flows, here and there on the cutting plane (see **Fig. 19**). Then, we can examine thereturn map on the Poincaré section to find any structure of the flows.If the flows take a complicated structure such as fractal one, however,the structure of the flows appear in different

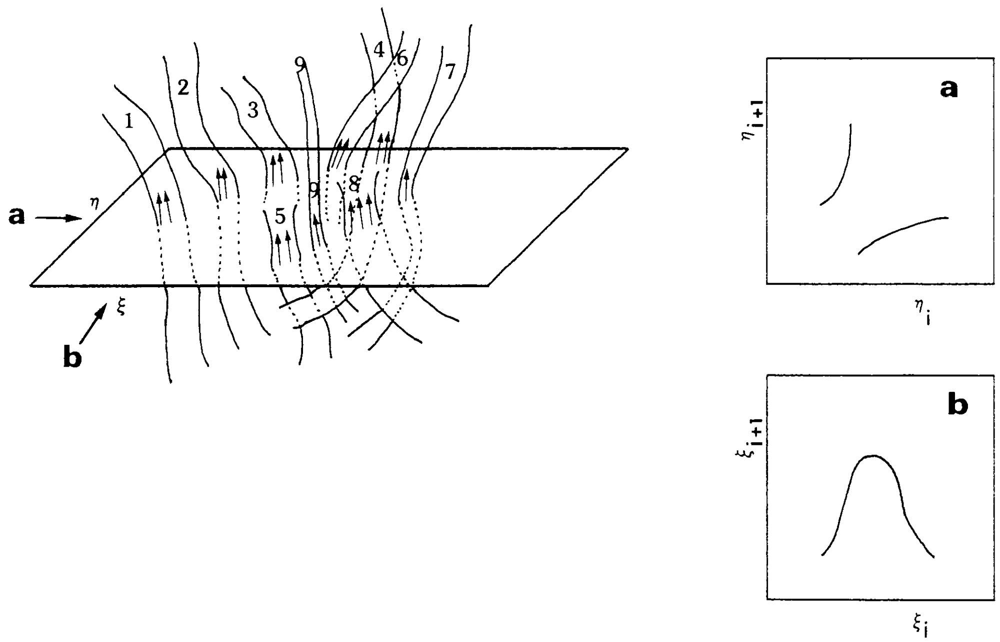

Fig. 19 Schematic Picture of Complicated Flows. The flows shown in this figure cause different features of return maps on a Poincaré section. The flows, which are divided into many subflows in the figure, have a fractal structure.

features, depending on a variable to be adopted to make a return map. Probably, the complicated structure of flows can produce ambivalent situation in the classification of the return maps. Such a case is schematically represented in Fig. 19.

Now we turn our discussion to dynamics of economic variables. The embedded manifold of (PEI, GDP, PHI) for period II shows one-dimensional feature in the return maps of intersecting points on the Poincare section at PHI=0.05 much clearer than other periods (see Fig. 9). It is natural to ask whether intersecting points of other cutting planes have the same nature of return maps or not. In **Figs. 20** and **21**, we show the return maps of intersecting points on other cutting planes for the same embedded manifold as that in Fig. 9. As seen in Figs. 20 and 21, one-dimensional feature of the return maps disappear. We interpret this fact as follows: Dynamics of economics is controlled by a lot of economic variables. However, if the temporal development of the growth rates is controlled by a few essential economic variables in a certain period, macro-control is applied to actual economics. This control may cause degeneration of behavior of economic variables. In an actual fact, it is considered that the Japanese macro-control may bring a situation where the dynamics of economics can be described by a few degrees of freedom of economic variables. If the growth rates of the essential economic variables lie in high-valued region, the control become loose. Hence each economic variables behave rather freely so that behavior of eco-

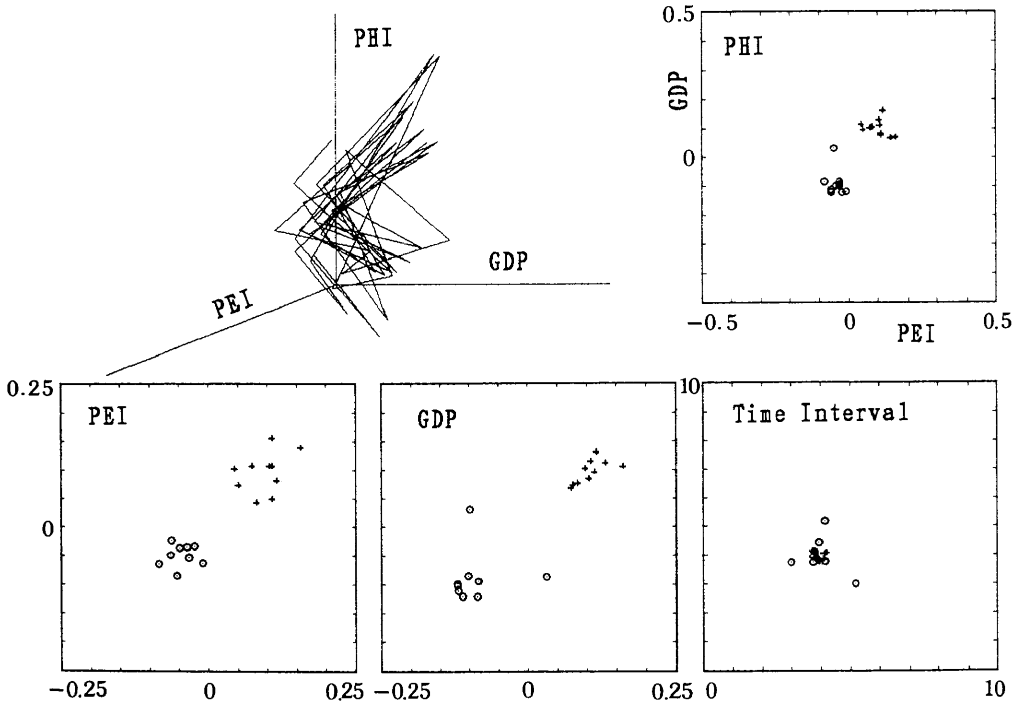

Fig. 20 The Same as Fig. 9, but PHI is set to 0.0.

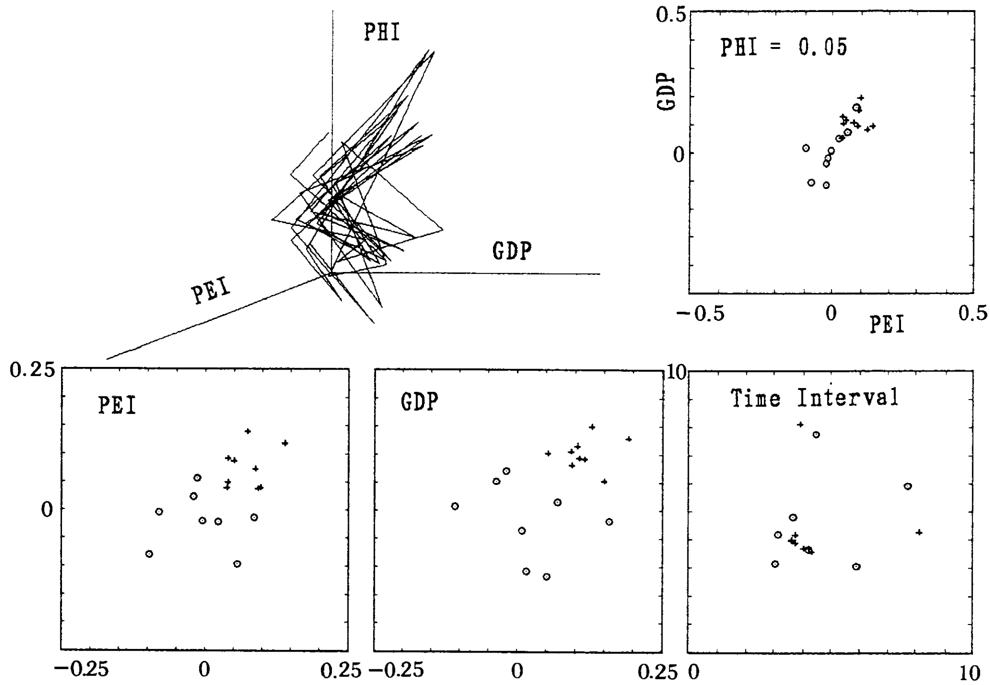

Fig. 21 The Same as Fig. 9, but PHI is set to −0.05.

nomic variables appears more stochastic (they are considered to be shifted to the deterministic chaos of higher degrees of freedom). This discussion can be described in a mathematical formulation as below.

Let X be the vector of economic variables of n components. By the observation of economics at the time t, we obtain only a part of them. It is represented by economic vector $Y(t)$ of m components ($m \leqq n$).We assume that the observed economic vector can be expressed as a transformation of the underlying economic variables $X(t)$;

$$Y(t)=TX(t). \tag{4.10}$$

T denotes a transformation matrix with size $m \times n$. Let the underlying economic variables expressed as the vector $X(t)$ be governed by the following ordinary differential equation,

$$\mathrm{d}X(t)/\mathrm{d}t=F(X(t)). \tag{4.11}$$

This ordinary differential equation is nonlinear in general. Therefore it is hard to solve it exactly. Using the following expansion of $F(X(t))$

$$F(X(t))=A+(\nabla\otimes F(X(t)))\cdot X(t)+1/2!(\nabla\otimes(\nabla\otimes F(X(t)))\cdot X(t))\cdot X(t)$$
$$+1/3!(\nabla\otimes(\nabla\otimes(\nabla\otimes F(X(t)))\cdot X(t))\cdot X(t))\cdot X(t)+\cdots, \tag{4.12}$$

the temporal development of the observed economic variables can be expressed as follows;

$$\mathrm{d}Y(t)/\mathrm{d}t=\mathrm{d}TX(t)/\mathrm{d}t=TF(X(t))=TA+T(\nabla\otimes F(X(t)))\cdot X(t)$$
$$+1/2!T(\nabla\otimes(\nabla\otimes F(X(t)))\cdot X(t))\cdot X(t)+\cdots, \tag{4.13}$$

where $\otimes$ denotes the operator of Kronecker product [15]. If one can obtain the right hand side of eq. (4.13) in a closed form of $Y(t)$, then the dynamics of the observed economic variables can be solved. However, it is usually hard to do so.

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