

## Original Paper

# Extinction and Growing-up of Solutions of Some Nonlinear Parabolic Equations

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**Synopsis:** This paper is concerned with the limiting behavior of solutions of the initial-boundary value problem for

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u \quad (1 < p < 2 \text{ and } \lambda > 0).$$

We show that solutions of this problem extinguish within a finite time for small initial data and grow up to infinity for large initial data. Moreover, we prove that there exist bounded solutions which exhibit some kind of “borderline” behavior.

We also treat the problem for  $u_t = \Delta u - u^{p-1} + \lambda u$  which indicates the another type of extinction.

## 1. Introduction

We are concerned here with the limiting behavior, including extinction and growing-up, of solutions of the initial-boundary value problem for the nonlinear singular parabolic equation:

$$(P1) \quad \begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u & x \in \Omega, \quad t > 0 & (1) \\ u(x, t) = 0 & x \in \partial\Omega, \quad t > 0 & (2) \\ u(x, 0) = \mu u_0(x) & x \in \Omega & (3) \end{cases}$$

where  $\Omega$  is a bounded domain in  $R^N$  with smooth boundary  $\partial\Omega$ ,  $1 < p < 2$ ,  $\lambda > 0$  and  $\mu > 0$ .

Here extinction means that solutions go to zero within a finite time and growing-up means that solutions go to infinity as time  $t$  tends to infinity.

For the problem (P1) with  $\lambda = 0$ , it is an easy matter to show that, for any initial data, there exists a finite number  $t^* > 0$  such that  $u(x, t) \rightarrow 0$  as  $t \rightarrow t^*$  and  $u(x, t) \equiv 0$  for  $t \geq t^*$ .

In case  $\lambda > 0$ , however, the term  $\lambda u$  prevents solutions going to zero.

The main purpose of this paper is to show extinction of solutions of (P1) for small initial data, growing-up of solutions for large initial data and, moreover, to show that there exist solutions which exhibit some kind of “borderline” behavior, that is, they neither extinguish within a finite time nor grow up to infinity.

The differential operator  $B(u) = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$  is homogeneous of degree  $p-1$ , that is,

$$B(ku) = k^{p-1}B(u) \quad \text{for } k \geq 0.$$

This property plays an essential role in our theory. The operator  $C(u) = -\Delta(|u|^{p-2}u)$  which appears in the problem of filtration and plasma physics is also homogeneous of

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degree  $p-1$ . Then one can apply our theory to this problem. We shall give some remarks on this problem in section 5.

The quasi-linear parabolic equation (1) with any  $p > 1$  has been actively studied and is a model for a broad class of singular and degenerate parabolic equations.

Existence and regularity results can be found in Lions [11], DiBenedetto [6] and Fukuda [8].

Especially, Neumann boundary value problem with  $\lambda = 0$ , that is, we take the boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega, \quad t > 0 \quad (2)'$$

instead of Dirichlet condition [2], has been treated by Alikakos-Evans [1], Alikakos-Rostamian [2], [3], [4] ( $p > 2$ ) and by Fukuda [8] ( $1 < p < 2$ ).

In their paper, decay estimates for the gradient of solutions in  $L^p(\Omega)$  ( $p > 2$ ) and  $L^\infty(\Omega)$  have been obtained ( $L^\infty$ -estimate under the assumption that  $\Omega$  is convex). Moreover, the regularizing effect has been proved using the monotonicity of  $\nabla u$  in  $L^\infty(\Omega)$ .

In [8], the homogenization of solutions for  $1 < p < 2$  has been proved, that is, there exists a finite number  $t^* > 0$  such that

$$u(x, t) \rightarrow \frac{1}{|\Omega|} \int u_0(x) dx \quad \text{as } t \rightarrow t^*$$

and

$$u(x, t) \equiv \frac{1}{|\Omega|} \int u_0(x) dx \quad \text{for } t \geq t^*.$$

In Section 2, we give the preliminary results including extinction of solutions of (P1) with  $\lambda = 0$ . Section 3 contains main results and theorems are then proved in Section 4. In Section 5, we give some remarks on the relation between problem (P1) and its stationary problem. We also refer to equations of porous medium type.

In Appendix, we discuss the another type of extinction. Semilinear parabolic equation

$$u_t = \Delta u - u^p + \lambda u$$

is treated there.

Throughout this paper, we denote  $L^p$ -norm  $\|\cdot\|_{L^p(\Omega)}$  by  $\|\cdot\|_p$ ,  $L^2$ -inner product by  $(\cdot, \cdot)$  and abbreviate  $\Omega$  in the integral  $\int_\Omega \cdot dx$ .

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## 2. Preliminaries

First of all, we present existence and extinction results for case  $\lambda = 0$ . Consider the initial-boundary value problem

$$(P2) \quad \begin{cases} v_\tau = \operatorname{div}(|\nabla v|^{p-2} \nabla v) & x \in \Omega, \quad \tau > 0 \\ v(x, \tau) = 0 & x \in \partial\Omega, \quad \tau > 0 \\ v(x, 0) = \mu u_0(x) & x \in \Omega. \end{cases} \quad \begin{matrix} (4) \\ (5) \\ (6) \end{matrix}$$

**Proposition 1** Let  $u_0$  be in  $L^{k+1}(\Omega)$ . ( $k \geq 1$ ) Then there exists a unique solution of (P2) satisfying

$$\begin{aligned} (1) \quad & v \in C(0, T; L^{k+1}(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega)) \\ (2) \quad & \tau^{1/2} v_\tau \in L^2(\Omega \times (0, T)). \end{aligned}$$

This is shown in [8] for the case of Neumann problem. An adaptation of the argument in [8] yields the result of Dirichlet problem.

**Proposition 2** Let  $\frac{2N}{N+2} \leq p < 2$  (if  $N \geq 3$ ) or  $1 < p < 2$  (if  $N = 1$  or  $2$ ) and  $v(x, \tau)$  be a solution of (P2) with initial data  $u_0 \in L^2(\Omega)$ .

Then there exists a finite positive number  $\tau^* (\equiv \tau^*(\mu u_0))$  depending on  $u_0$  and  $\mu$  such that

$$v(\cdot, \tau) \rightarrow 0 \quad \text{in } L^2(\Omega) \quad \text{as } \tau \rightarrow \tau^*$$

and

$$v(x, \tau) \equiv 0 \quad \text{for } \tau \geq \tau^*, \quad x \in \Omega$$

where  $\tau^*$  is bounded above by  $\frac{1}{(2-p)K^p} \|\mu u_0\|_2^{2-p}$  and  $K$  is a constant in lemma 1 which is mentioned below.

**Remark** Proposition 2 assures that  $T$  in Proposition 1 can be replaced by  $\infty$ , that is, the solution  $v(x, \tau)$  of (P2) continues globally up to  $\infty$ .

Before proof of Proposition 2, we give some elementary lemmas.

**Lemma 1** (Sobolev–Poincaré’s inequality) If  $\frac{2N}{N+2} \leq p < 2$  ( $N \geq 3$ ) and  $u$  belongs to  $W_0^{1,p}(\Omega)$ , then

$$K \|u\|_2 \leq \|\nabla u\|_p$$

where  $K$  is a positive constant independent of  $u$ .

**Lemma 2** (Gagliardo–Nirenberg type inequality) Let  $u$  belong to  $W_0^{1,2}(\Omega) \cap L^p(\Omega)$  ( $1 < p < 2$ ). Then

$$K_0 \|u\|_2^v \leq \|\nabla u\|_2^2 + \|u\|_p^p$$

where  $p < v < 2$  and  $K_0$  is a positive constant independent of  $u$ .

For simplicity, we give the formal proof only.

**Proof of Proposition 2** Multiplication (4) by  $v(x, \tau)$  and integration by parts yield

$$\frac{1}{2} \frac{d}{d\tau} \int v(x, \tau)^2 dx + \int |\nabla v(x, \tau)|^p dx = 0. \quad (7)$$

From lemma 1, we have

$$\frac{d}{d\tau} \|v(\tau)\|_2 + K^p \|v(\tau)\|_2^{p-1} \leq 0, \quad (8)$$

which implies

$$\|v(\tau)\|_2 \leq (\|\mu u_0\|_2^{2-p} - K^p(2-p)\tau)^{1/(2-p)}$$

Hence as

$$\tau \rightarrow \tau^* \leq \frac{1}{(2-p)K^p} \|\mu u_0\|_2^{2-p}, \quad \|v(\tau)\|_2 \rightarrow 0.$$

**Remarks (1)** To prove Proposition 2 rigorously, one can use the approximate problem:

$$(P2)_\varepsilon \quad \begin{cases} v_\tau^\varepsilon = \operatorname{div} ((|\nabla v^\varepsilon|^2 + \varepsilon)^{(p-2)/2} \nabla v^\varepsilon) & x \in \Omega, \quad \tau > 0, \\ v^\varepsilon(x, \tau) = 0 & x \in \partial\Omega, \quad \tau > 0, \\ v^\varepsilon(x, 0) = \mu u_0^\varepsilon(x) & x \in \Omega \end{cases}$$

(2) In Proposition 2, we can avoid the restriction on  $p$  as follows:

**Proposition 3** Let  $u_0$  belongs to  $L^k(\Omega)$  where  $k \geq \frac{N(2-p)}{p}$ . Then there exists a finite positive number  $\tau^*$  such that

$$v(\cdot, \tau) \rightarrow 0 \quad \text{in } L^k(\Omega) \quad \text{as } \tau \rightarrow \tau^*$$

and

$$v(x, \tau) \equiv 0 \quad \text{for } \tau \geq \tau^*, \quad x \in \Omega.$$

### 3. Main results

As was mentioned in section 1, we may consider three possibilities on the asymptotic states of problem (P1).

(I) There exists a finite number  $t^* > 0$  such that

$$\|u(t)\|_2 \rightarrow 0 \quad \text{as } t \rightarrow t^*$$

and

$$u(x, t) \equiv 0 \quad \text{for } t \geq t^*, \quad x \in \Omega.$$

(II) There exist two positive constants  $m$  and  $M$  such that

$$m \leq \|u(t)\|_2 \leq M \quad \text{for any } t \geq 0.$$

(III) Solutions grow up to infinity as  $t$  tends to  $\infty$ .

The main theorems are as follows:

**Theorem 1** Let  $\frac{2N}{N+2} \leq p < 2$  ( $N \geq 3$ ) or  $1 < p < 2$  ( $N = 1$  or  $2$ ),  $\alpha = \frac{1}{\lambda(2-p)}$ ,  $u(x, t)$  be

a solution of (P1) and  $\tau^*$  be an extinction time of  $v(x, \tau)$  which is a solution of (P2) where  $u(x, t) = e^{\lambda t} v(x, \tau)$  and  $\tau = \frac{1}{\lambda(2-p)} (1 - e^{-\lambda(2-p)t})$ .

Then if  $\tau^* < \alpha$ , (I) holds and  $t^* = \alpha \log \left( \frac{\alpha}{\alpha - \tau^*} \right)$ . If  $\tau^* = \alpha$ , (II) holds. If  $\tau^* > \alpha$ , (III) holds and

$$e^{-\lambda t} \|u(t)\|_2 \longrightarrow \|v(\alpha)\|_2 \quad \text{as } t \longrightarrow \infty.$$

**Theorem 2** Under the same assumption as Theorem 1. Let  $s^*(u_0)$  be an extinction time for the problem (P2) with  $\mu = 1$ , and  $\beta = \left( \frac{\alpha}{s^*(u_0)} \right)^{1/(2-p)}$ .

Then, if  $\mu < \beta$ , (I) holds. If  $\mu = \beta$ , (II) holds. If  $\mu > \beta$ , (III) holds.

Now we consider the problem (P1) with  $\mu = 1$  and classify the limiting behavior by the size of  $u_0$ .

**Theorem 3** Let  $\frac{2N}{N+2} \leq p < 2$  ( $N \geq 3$ ) and  $u(x, t)$  be a solution of (P1) with  $\mu = 1$  and  $u_0 \in L^2(\Omega) \cap W_0^{1,p}(\Omega)$ .

If  $\|u_0\|_2 < \left( \frac{K^p}{\lambda} \right)^{1/(2-p)}$ , then  $\tau^* < \alpha$  and hence (I) holds.

If  $\|u_0\|_2^2 > \frac{2}{\lambda p} \|\nabla u_0\|_p^p$ , then  $\tau^* > \alpha$  and hence (III) holds.

**Remark**  $\|u_0\|_2^2 > \frac{2}{\lambda p} \|\nabla u_0\|_p^p$  implies  $\|u_0\|_2 < \left( \frac{K^p}{\lambda} \right)^{1/(2-p)} \left( \frac{2}{p} \right)^{1/(2-p)}$  and then there is a gap between the case (I) and (III).

#### 4. Proofs of Theorems

To avoid the complexity of calculation, we shall give the idea of proof and prove theorems formally. To make them rigorous, one can use the approximate problem (P2) <sub>$\epsilon$</sub>  adding the term  $\lambda u^\epsilon$ .

**Proof of theorem 1** Let  $u(x, t) = e^{\lambda t} v(x, \tau)$  and

$$\tau = \frac{1}{\lambda(2-p)} (1 - e^{-\lambda(2-p)t}). \quad (9)$$

Then one can reduce the problem (P1) into (P2). Proposition 2 assures that solutions  $v(x, \tau)$  of (P2) extinguish within a finite time  $\tau^*$  for any initial data belonging to  $L^2(\Omega)$ .

One can find from (9) that as  $t$  varies from 0 to  $\infty$ ,  $\tau$  varies from 0 to  $\alpha \left( = \frac{1}{\lambda(2-p)} \right)$ .

Hence we have to discuss three cases.

First case: when  $\tau^*$  is less than  $\alpha$ ,  $t$  tends to  $t^* \left( = \alpha \log \left( \frac{\alpha}{\alpha - \tau^*} \right) \right)$  as  $\tau$  tends to  $\tau^*$ , and then

$\|v(\tau)\|_2$  goes to zero and also  $\|u(t)\|_2$  extinguishes as  $t \rightarrow t^*$ .

Second case: when  $\tau^*$  is larger than  $\alpha$ . Whenever  $t$  varies from 0 to  $\infty$ ,  $\tau$  can not reach  $\tau^*$ .  $\tau$  only reaches  $\alpha$ . Hence as  $\tau$  tends to  $\alpha$ ,  $t$  goes to  $\infty$  and

$$\|v(\tau)\|_2 = e^{-\lambda t} \|u(t)\|_2 \longrightarrow \|v(\alpha)\|_2.$$

That is,  $u(x, t)$  behaves like as  $e^{\lambda t} v(x, \alpha)$  near  $t = \infty$ .

Last case: when  $\tau^*$  is equal to  $\alpha$ . We can apply the similar method as in Berryman-Holland [5] to problem (P2).

Dividing (8) by  $\|v(\tau)\|_2^{p-1}$  and integrating this from  $\tau$  to  $\sigma$  ( $0 \leq \tau < \sigma < \tau^* = \alpha$ ), we have

$$\|v(\sigma)\|_2^{2-p} - \|v(\tau)\|_2^{2-p} + (2-p)K^p(\sigma - \tau) \leq 0.$$

Letting  $\sigma$  tend to  $\alpha$ , then  $\|v(\sigma)\|_2$  tends to zero and

$$\|v(\tau)\|_2 \geq \{(2-p)K^p(\alpha - \tau)\}^{1/(2-p)}$$

Using  $v(x, \tau) = e^{-\lambda t} u(x, t)$  and  $\left(\frac{\alpha - \tau}{\alpha}\right)^{1/(2-p)} = e^{-\lambda t}$ , we have

$$\|u(t)\|_2 \geq \{(2-p)\alpha K^p\}^{1/(2-p)} = m > 0.$$

Next we shall show the upper estimate of  $\|u(t)\|_2$ .

Integration by parts and the Cauchy-Schwarz inequality yield

$$\left(\int |\nabla v|^p dx\right)^2 = \left(-\int \Delta_p v \cdot v dx\right)^2 \leq \int |\Delta_p v|^2 dx \int v^2 dx.$$

For simplicity we denote  $\text{div}(|\nabla v|^{p-2} \nabla v)$  as  $\Delta_p v$  and suppress the variables  $x$  and  $\tau$  in the integrand.

It is convenient to rewrite this as

$$\frac{\int |\nabla v|^p dx}{\int v^2 dx} \leq \frac{\int |\Delta_p v|^2 dx}{\int |\nabla v|^p dx}$$

An expression for  $\frac{1}{p} \frac{d}{d\tau} \int |\nabla v|^p dx$  can be derived as follows

$$\frac{1}{p} \frac{d}{d\tau} \int |\nabla v|^p dx = - \int |\Delta_p v|^2 dx.$$

Using this expression and (7), we have

$$\frac{\frac{1}{2} \frac{d}{d\tau} \int v^2 dx}{\int v^2 dx} \geq \frac{\frac{1}{p} \frac{d}{d\tau} \int |\nabla v|^p dx}{\int |\nabla v|^p dx}$$

which implies

$$\frac{\|\nabla v(\sigma)\|_p}{\|\nabla v(\tau)\|_p} \geq \frac{\|v(\sigma)\|_2}{\|v(\tau)\|_2} \quad (10)$$

for  $0 \leq \sigma < \tau \leq \tau^*$ .

We define  $g(\tau) = \|v(\tau)\|_2^{2-p}$ . Then (10) implies that

$$g'(\sigma) \leq g'(\tau) \quad \text{for any } \sigma \leq \tau. \quad (11)$$

Integrating (11) from  $\sigma$  to  $T$  with respect to  $\sigma$  and from  $\tau$  to  $T$  with respect to  $\tau$ , we obtain

$$\frac{g(T) - g(\tau)}{T - \tau} \geq \frac{g(T) - g(\sigma)}{T - \sigma} \quad 0 \leq \sigma \leq \tau \leq T.$$

Letting  $T$  tend to  $\tau^*$  and  $\sigma$  to 0, we have

$$g(\tau) \leq \left(1 - \frac{\tau}{\tau^*}\right) g(0).$$

This inequality together with  $v(x, \tau) = e^{-\lambda \tau} u(x, \tau)$  and  $1 - \frac{\tau}{\tau^*} = e^{-(2-p)\lambda \tau}$  yields

$$\|u(t)\|_2 \leq \|u_0\|_2 = M.$$

**Proof of Theorem 2** Let  $s = \mu^{p-2} \tau$  and  $v(x, \tau) = \mu w(x, s)$ . Then we can change (P2) to

$$(P3) \quad \begin{cases} w_s = \operatorname{div}(|\nabla w|^{p-2} \nabla w) & x \in \Omega, \quad s > 0, \\ w(x, s) = 0 & x \in \partial\Omega, \quad s > 0, \\ w(x, 0) = u_0(x) & x \in \Omega. \end{cases}$$

We denote the extinction time of (P3) by  $s^* = s^*(u_0)$ . Then we can express the relation between  $s^*$  and extinction time  $\tau^*$  of (P2) as follows

$$\tau^*(\mu u_0) = \mu^{2-p} s^*(u_0)$$

from which we can easily conclude Theorem 2.

**Proof of Theorem 3** It is easily obtained that the assumption  $\|u_0\|_2 < \left(\frac{K^p}{\lambda}\right)^{1/(2-p)}$  implies  $\tau^* < \alpha$  since  $\tau^*$  is bounded above by  $\frac{1}{(2-p)K^p} \|\mu u_0\|_2^{2-p}$ .

To prove  $\tau^* > \alpha$ , we begin by getting the lower estimate of  $\|v(\tau)\|_2$ . We multiply (1) by  $u_t$  to yield

$$\int u_t^2 dx + \frac{d}{dt} \left\{ \frac{1}{p} \int |\nabla u|^p dx - \frac{\lambda}{2} \int u^2 dx \right\} = 0. \quad (12)$$

Since the first term of (12) is non-negative and  $\|u_0\|_2^2 > \frac{2}{\lambda p} \|\nabla u_0\|_p^p$ , we have

$$\|\nabla u(t)\|_p^p < \frac{\lambda p}{2} \|u(t)\|_2^2.$$

We can rewrite this to

$$\|\nabla v(\tau)\|_p^p < \frac{\lambda p}{2} e^{(2-p)\lambda \tau} \|v(\tau)\|_2^2 = \frac{\lambda p}{2(1-(2-p)\lambda \tau)} \|v(\tau)\|_2^2.$$

This inequality and (7) implies

$$\frac{d}{d\tau} \int v^2 dx + \frac{\lambda p}{1-(2-p)\lambda \tau} \int v^2 dx > 0$$

which implies

$$\|v(\tau)\|_2^2 > \|u_0\|_2^2 (1 - (2-p)\lambda\tau)^{p/(2-p)}$$

This means that  $\|v(\tau)\|_2$  does not extinguish for  $t \in (0, \alpha]$ , Then  $\tau^* > \alpha$  and (III) holds.

## 5. Remarks

In theorem 2, we have shown that there exist solutions of (P1) which exhibit some kind of “borderline” behavior, that is, they neither extinguish within a finite time nor grow up to infinity. We are interested in the asymptotic behavior of these solutions which we call bounded solutions.

(1) Now we discuss stationary states of the problem (P1). Consider the elliptic problem:

$$(EP) \quad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda u & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \quad (13)$$

Otani has proved existence and nonexistence of nontrivial solutions of (EP).

**Theorem 4** (Otani [13]) If  $\frac{2N}{N+2} < p < 2$  ( $N \geq 2$ ), then there exists at least one nontrivial non-negative solutions of (EP) belonging to  $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ .

If  $1 < p < \frac{2N}{N+2}$  ( $N \geq 3$ ) and  $\Omega$  is star shaped, then there are no nontrivial solutions of (EP) belonging to  $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ .

If  $p = \frac{2N}{N+2}$  ( $N \geq 3$ ) and  $\Omega$  is strictly star shaped, then there are no nontrivial solutions of definite sign belonging to  $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ .

Here  $\Omega$  is said to be star shaped (resp. strictly star shaped), if  $n(x) \cdot x \geq 0$  (resp.  $n(x) \cdot x \geq \rho > 0$ ) holds for all  $x \in \partial\Omega$  with a suitable choice of the origin, where  $n(x)$  denotes the outward normal unit vector at  $x \in \partial\Omega$ .

He also proved that there exists a unique positive solution of (EP) for  $N=1$ . [12]

(2) Combining the above results and theorem 2, we can expect that if  $\frac{2N}{N+2} < p < 2$  ( $N \geq 3$ ) or  $1 < p < 2$  ( $N=1$  or  $2$ ) a bounded solution tends to a nontrivial solution as  $t$  tends to  $\infty$ . However, in case  $p = \frac{2N}{N+2}$ , the curious phenomena happen that there exists a bounded solution which neither extinguish within a finite time nor grow up to infinity, but there are no nontrivial solutions of definite sign of (EP) where  $\Omega$  is strictly star shaped.

We shall discuss on the details in the forthcoming paper.

(3) If we take a nontrivial solution of (EP) as an initial value  $u_0$ , it is easily found that  $\tau^* = \alpha$  and  $\mu = 1$ .



Nontrivial solutions of (EP) are unstable as initial values, that is, let  $U(x)$  be a nontrivial non-negative solution of (EP) and  $u_0(x) = kU(x)$ . If  $k < 1$  then  $u(x, t; u_0)$  extinguishes within a finite time and if  $k > 1$  then  $u(x, t; u_0)$  grows up to infinity.

On the other hand, if we take  $u_0(x) = V(x)$  which is not a nontrivial solution of (EP), theorem 2 assures that there exist a  $\mu > 0$  such that a solution  $u(x, t; \mu V(x))$  of (P1) is a bounded solution.

From the above results we can say that there exists a positive constant  $\mu$  and a  $L^2(\Omega)$ -function  $V(x)$  such that  $u(x, t; \mu V(x))$  is a bounded solution of (P1) and  $U(x) - \mu V(x)$  is not definite sign all over  $\Omega$  for a nontrivial solution  $U(x)$  of (EP).

(4) We discuss briefly here on the equation of porous medium type.

Consider the problem

$$(P4) \quad \begin{cases} u_t = \Delta(u^m) + \lambda u & x \in \Omega, \quad t > 0, \\ u(x, t) = 0 & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = \mu u_0(x) & x \in \Omega \end{cases}$$

where  $\lambda > 0$ ,  $0 < m < 1$  and  $\mu > 0$ .

In case  $m > 1$  and  $\lambda = 0$ , the equation  $u_t = \Delta(u^m)$  is often called the porous medium equation since it first arose in the study of gas flows in homogeneous porous media [10].

Problems with  $m > 1$  and  $\lambda > 0$  arise in the Gurtin-MacCamy theory of density dependent diffusion of biological populations [9].

The operator  $C(u) = -\Delta(u^m)$  is homogeneous of degree  $m$ . Then if  $0 < m < 1$ , the same argument as above can be applied to Problem (P4) under the smoothness assumption like as in Berrymann-Holland [5].

## 6. Appendix

We discuss here on the another type of extinction phenomena.

Consider the initial-boundary value problem of a semilinear parabolic equation

$$(SP) \quad \begin{cases} u_t = \Delta u - u^{p-1} + \lambda u & x \in \Omega, \quad t > 0, & (A \cdot 1) \\ u(x, t) = 0 & x \in \partial\Omega, \quad t > 0, & (A \cdot 2) \\ u(x, 0) = u_0(x) \geq 0 & x \in \Omega & (A \cdot 3) \end{cases}$$

where  $\Omega$  is a bounded domain in  $R^N$  with smooth boundary  $\partial\Omega$ ,  $\lambda \geq 0$  and  $1 < p < 2$ .

In case  $\lambda = 0$ , it is well-known that solutions of (SP) with  $\lambda = 0$  extinguish within a finite time. Evans and Knerr [7] established this for the Cauchy problem by constructing a suitable comparison function.

Now let  $u(x, t)$  be a  $L^\infty$ -solution of (SP),  $\lambda_1$  be the first eigenvalue of  $-\Delta$  with zero Dirichlet boundary condition and  $\phi$  be an eigenfunction corresponding to  $\lambda_1$ .

We have the following:

**Theorem A. 1** If  $\lambda \leq \lambda_1$ , then there exists a finite number  $t^* > 0$  such that

$$u(\cdot, t) \longrightarrow 0 \quad \text{in } L^2(\Omega) \quad \text{as } t \longrightarrow t^*$$

for any size of initial data.

**Theorem A. 2** Let  $\delta = \frac{(2-\nu)K_0}{(2-p)\lambda}$  where  $\nu$  is a constant in Lemma 2. If  $\lambda_1 < \lambda$  and  $\|u_0\|_2 < \delta^{1/(2-\nu)}$  then there exists a finite number  $t^* > 0$  such that

$$u(\cdot, t) \rightarrow 0 \quad \text{in } L^2(\Omega) \quad \text{as } t \rightarrow t^* \leq \alpha \log \left\{ \frac{\delta}{\delta - \|u_0\|_2^{2-\nu}} \right\}.$$

**Theorem A. 3** If  $\lambda_1 < \lambda$  and  $(u_0, \phi) > (\lambda - \lambda_1)^{-1/(2-p)}$ , then

$$\|u(t)\|_\infty \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

Moreover

$$\|u(t)\|_\infty \geq \text{Const. } e^{(\lambda - \lambda_1)t}.$$

When  $\lambda < \lambda_1$ , the term  $\Delta u$  dominates the term  $\lambda u$ , that is, solutions of (SP) behave as same as those of (SP) with  $\lambda = 0$ . Then all solutions of (SP) in this case extinguish within a finite time for any size of initial data.

The proof of Theorem A. 1 is evident. We omit it.

**Proof of Theorem A. 2** By the change of dependent variables such that

$$u(x, t) = e^{\lambda t} v(x, t)$$

problem (SP) can be transformed to

$$(SP)_1 \quad \begin{cases} v_t = \Delta v - e^{-(2-p)\lambda t} v^{p-1} & x \in \Omega, \quad t > 0, \\ v(x, t) = 0 & x \in \partial\Omega, \quad t > 0, \\ v(x, 0) = u_0(x) & x \in \Omega. \end{cases}$$

Multiply this equation by  $v(x, t)$  and integrate by parts to find

$$\frac{1}{2} \frac{d}{dt} \int v(x, t)^2 dx + e^{-(2-p)\lambda t} \left\{ \int |\nabla v(x, t)|^2 dx + \int v(x, t)^p dx \right\} \leq 0.$$

Lemma 2 implies

$$\frac{d}{dt} \|v(t)\|_2 + K_0 e^{-(2-p)\lambda t} \|v(t)\|_2^{p-1} \leq 0.$$

From this differential inequality, we have

$$\|v(t)\|_2 \leq \{(\|u_0\|_2^{2-\nu} - \delta) + \delta e^{-(2-p)\lambda t}\}^{1/(2-\nu)}$$

If  $\|u_0\|_2 < \delta^{1/(2-\nu)}$ , then  $\|v(t)\|_2 \rightarrow 0$  as  $t \rightarrow t^*$ , from which we conclude Theorem A. 2.

**Proof of Theorem A. 3** We assume  $\phi > 0$  in  $\Omega$  and normalize  $\phi$  such that  $\int_\Omega \phi(x) dx = 1$ .

Multiply (A. 1) by  $\phi(x)$  and integrate over  $\Omega$  to find

$$\frac{d}{dt} (u(t), \phi) = (\Delta u(t), \phi) + \lambda (u(t), \phi) - (u(t)^{p-1}, \phi).$$

Using integration by parts and the equation  $-\Delta \phi = \lambda_1 \phi$ , we have

$$\frac{d}{dt}(u(t), \phi) - (\lambda - \lambda_1)(u(t), \phi) + (u(t))^{p-1}, \phi = 0.$$

Now, since  $1 < p < 2$ , Jensen's inequality implies

$$\frac{d}{dt}(u(t), \phi) - (\lambda - \lambda_1)(u(t), \phi) + (u(t), \phi)^{p-1} \geq 0$$

from which we have

$$(u(t), \phi) \geq e^{(\lambda - \lambda_1)t} \left\{ (u_0, \phi)^{2-p} - \left( \frac{1}{\lambda - \lambda_1} \right) \right\}^{1/(2-p)}$$

Then if  $(u_0, \phi) > \left( \frac{1}{\lambda - \lambda_1} \right)^{1/(2-p)}$ , we conclude Theorem A. 3 since  $\|u(t)\|_\infty \geq (u(t), \phi)$ .

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