論文　**Original Paper**

Irreducibilities of the induced characters of some p-groups

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Abstract: Let ϕ be a faithful irreducible character of the cyclic group C_n of order p^n, where p is an odd prime. We study the p-group G containing C_n such that the induced character ϕ^G is also irreducible. The purpose of this paper is to consider the subgroup $N_G(N_G(C_n))$. We will determine the factor group $N_G(N_G(C_n))/N_G(C_n)$.

Keywords: p-group, Extension, Irreducible induced character, Faithful irreducible character

1. Introduction

Let G be a finite group. We denote by Irr(G) the set of complex irreducible characters of G and by FIrr(G) ($\subset$ Irr(G)) the set of faithful irreducible characters of G.

Let p be a prime. For a non-negative integer n, we denote by C_n the cyclic group of order p^n. A finite group G is called an M-group, if every $\phi \in \mathrm{Irr}(G)$ is induced from a linear character of a subgroup of G.

It is well-known that every nilpotent group is an M-group. Hence, when G is a p-group, for any $\chi \in \mathrm{Irr}(G)$, there exists a subgroup H of G and a linear character ϕ of H such that $\phi^G = \chi$. If we set $N = \mathrm{Ker}\,\phi$, then $N \lhd H$ and ϕ is a faithful irreducible character of $H/N \cong C_n$, for some non-negative integer n. In this paper, we will consider the case when $N=1$, that is, ϕ is a faithful linear character of $H \cong C_n$.

We consider the following:

Problem 1. *Let p be an odd prime, and ϕ be a faithful irreducible character of C_n. Determine the p-group G such that $C_n \subset G$ and the induced character ϕ^G is also irreducible.*

Since all the faithful irreducible characters of C_n are algebraically conjugate to each other, the irreducibility of ϕ^G ($\phi \in \mathrm{FIrr}(C_n)$) is independent of the choice of ϕ, and depends only on n.

This problem has been solved in each of the following cases:

(1) $C_n \lhd G$ ([2]),

(2) G has a subgroup H containing C_n such that $C_n \lhd H$ and $[G : H] = p$ ([6]).

(3) $[G : H] = p^3$ ([7]).

On the other hand, when $p=2$, Yamada and Iida [4] proved the following interesting result:

Let $\mathbf{Q}$ denote the rational field. Let G be a 2-group and χ a complex irreducible character of G. Then there exist subgroups $H \rhd N$ in G and a complex irreducible character ϕ of H such that $\chi = \phi^G$, $\mathbf{Q}(\chi) = \mathbf{Q}(\phi)$, $N = \mathrm{Ker}\,\phi$ and

$H/N \cong Q_n (n \geq 2)$, or $D_n (n \geq 2)$,
or $SD_n (n \geq 3)$, or $C_n (n \geq 0)$.

Here, Q_n, D_n and SD_n denote the generalized quaternion group, the dihedral group of order 2^{n+1} ($n \geq 2$) and the semidihedral group of order 2^{n+1} ($n \geq 3$), respectively, and $\mathbf{Q}(\chi) = \mathbf{Q}(\chi(g), g \in G)$.

They considered the following:

Problem 2. *Let ϕ be a faithful irreducible character of H, where $H = Q_n$ or D_n or SD_n. Determine the 2-group G such that $H \subset G$ and the induced character ϕ^G is also irreducible.*

Yamada and Iida [3] solved this problem in the case when $[G; H] = 2$ or 4 and we have recently solved Problem 2 completely ([8]). In [8], we showed that

$$G = N_G(H) \text{ or } N_G(N_G(H)),$$

for all $H = Q_n$ or D_n or SD_n, if G satisfies the conditions of Problem 2. Here, as usual, $N_G(H)$ and $N_G(N_G(H))$ are the normalizers of H and $N_G(H)$ in G, respectively. This means that, if we define subgroups of G by

$M_1 = N_G(H)$, and $M_{i+1} = N_G(M_i)$, for $i \geq 1$,

then

$H \subsetneqq M_1 \subsetneqq M_2 = M_3 = M_4 = \cdots = G$,

for all $H = Q_n$ or D_n or SD_n. Concerning Problem 2, see also [5].

In this paper, we consider Problem 1. We also define subgroups of G by

$N_1 = N_G(C_n)$, and $N_{i+1} = N_G(N_i)$, for $i \geq 1$.

The purpose of this paper is to consider the group $N_2 = N_G(N_G(C_n))$. We will determine the structure of the group N_2/N_1.

Throughout this paper, $\mathbf{Z}$ and $\mathbf{N}$ denote the set of rational integers and the natural numbers, respectively.

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2. Statements of the results

For the rest of this paper, we assume that p is an odd prime.

First, we introduce the following groups:

(i) $G(n, m) = \langle a, b_m \rangle$ with
$a^{p^n} = b_m^{p^m} = 1$, $b_m a b_m^{-1} = a^{1+p^{n-m}}$, $(m \leqq n-1)$.

(ii) $G(n, m, 1) = \langle a, b_m, v \rangle (\triangleright G(n, m) = \langle a, b_m \rangle)$ with
$a^{p^n} = b_m^{p^m} = 1$, $b_m a b_m^{-1} = a^{1+p^{n-m}}$,
$vav^{-1} = a^{1+p^{n-m-1}} b_m^{p^{m-1}}$,
$v^p = b_m$, $v b_m v^{-1} = b_m$ $(2m \leqq n-1)$.

(iii) $G(n, 1, 1, 1) = \langle a, b_1, v, x \rangle (\triangleright G(n, 1, 1) = \langle a, b_1, v \rangle)$ with
$a^{p^n} = b_1^p = 1$, $b_1 a b_1^{-1} = a^{1+p^{n-1}}$, $vav^{-1} = a^{1+p^{n-2}} b_1$,
$v^p = b_1$, $v b_1 v^{-1} = b_1$, $xax^{-1} = a^{1+p^{n-3}} v$, $x^p = v$,
$xvx^{-1} = v$, $x b_1 x^{-1} = b_1$ $(7 \leqq n)$.

We can see that $G(n, m, 1)$ (resp. $G(n, 1, 1, 1)$) is an extension group of $G(n, m)$ (resp. $G(n, 1, 1)$) by using Proposition 1 below:

Proposition 1. *Let N be a finite group such that $G \triangleright N$ and $G/N = \langle uN \rangle$ is a cyclic group of order m. Then $u^m = c \in N$. If we put $\sigma(x) = uxu^{-1}$, $x \in N$, then $\sigma \in Aut(N)$ and (i) $\sigma^m(x) = cxc^{-1}$, $(x \in N)$ (ii) $\sigma(c) = c$.*

Conversely, if $\sigma \in Aut(N)$ and $c \in N$ satisfy (i) and (ii), then there exists one and only one extension group G of N such that $G/N = \langle uN \rangle$ is a cyclic group of order m and $\sigma(x) = vxv^{-1}$ $(x \in N)$ and $v^m = c$.

Proof. For instance, see [9, III, § 7].

The structure of the group $N_1 = N_G(C_n)$ was determined by Iida([2]).

Theorem 0.1 (Iida [2]) *Let G be a p-group which contains C_n as a normal subgroup of index p^m. Let $\phi \in \mathrm{FIrr}(C_n)$. Suppose that $\phi^G \in \mathrm{Irr}(G)$. Then $m \leqq n-1$, and $G \cong G(n, m)$.*

On the other hand, $N_2 = N_G(N_1)$ and $N_3 = N_G(N_2)$ were determined, when $[N_1 : C_n] = p$ ([7]).

Theorem 0.2 ([7]) *Let p be an odd prime. Let G be a p-group which contains $C_n = \langle a \rangle$. We assume that $[G : C_n] \geq p^3$. Define the subgroups of G by*
$N_1 = N_G(C_n)$, and $N_{i+1} = N_G(N_i)$, for $i = 1, 2$.
Let $\phi \in \mathrm{FIrr}(C_n)$. Suppose that $\phi^G \in \mathrm{Irr}(G)$, and $[N_1 : C_n] = p$. Then
(1) $N_2/N_1 \cong C_1$ and $N_2 \cong G(n, 1, 1)$,
(2) $N_3/N_2 \cong C_1$ and $N_3 \cong G(n, 1, 1, 1)$.

REMARK 1. Conversely, it is easy to see that the groups $G(n, 1, 1)$ and $G(n, 1, 1, 1)$ satisfy the condition (EX, C), which is defined in section 3 of this paper. Hence these groups satisfy the conditions of Problem 1.

REMARK 2. By results of Iida ([2], see Theorem 0.1. in this paper), we can see that $N_1 \cong G(n, 1)$.

Our main theorem is the following:

Theorem. *Let p be an odd prime. Let G be a p-group which contains $C_n = \langle a \rangle$. Define the subgroups of G by*
$N_1 = N_G(C_n)$, and $N_2 = N_G(N_1)$.
Let $\phi \in \mathrm{FIrr}(C_n)$. Suppose that $\phi^G \in \mathrm{Irr}(G)$, and $[N_1 : C_n] = p^m$, $4m \leqq n$. Then
$N_2/N_1 \cong C_t$ where $t \leqq m$.

3. Some preleminary results

In this section, we state some results concerning the criterion of the irreducibilities of induced characters and others, which we need in section 4.

We denote by $\zeta = \zeta_{p^n}$ a primitive p^nth root of unity. It is known that, for $C_n = \langle a \rangle$, there are p^n irreducible characters ϕ_ν $(1 \leqq \nu \leqq p^n)$ of C_n:
$\phi_\nu(a^i) = \zeta^{\nu i}$, $(1 \leqq i \leqq p^n)$.
The irreducible character ϕ_ν is faithful if and only if $(\nu, p) = 1$.
It is well-known that
$\mathrm{Aut}\langle a \rangle \cong (\mathbf{Z}/p^n\mathbf{Z})^* \cong C_* \times C_{n-1}$

where $(\mathbf{Z}/p^n\mathbf{Z})^*$ is the unit group of the factor ring $\mathbf{Z}/p^n\mathbf{Z}$ and C_* is the cyclic group of order $p-1$. Further, C_{n-1} is generated by the element $1+p$ in $\mathbf{Z}/p^n\mathbf{Z}$.

First, we state the following result of Shoda (cf [1, p. 329]):

Proposition 2. *Let G be a group and H be a subgroup of G. Let ϕ be a linear character of H. Then the induced character ϕ^G of G is irreducible if and only if, for each $x \in G - H = \{g \in G \mid g \notin H\}$, there exists $h \in xHx^{-1} \cap H$ such that $\phi(h) \neq \phi(xhx^{-1})$. (Note that, when ϕ is faithful, the condition $\phi(h) \neq \phi(xhx^{-1})$ holds if and only if $h \neq xhx^{-1}$).*

Using this result, we have the following:

Proposition 3. *Let $\langle a \rangle = C_n \subset G$, and ϕ be a faithful irreducible character of C_n. Then the following conditions are equivalent:*
(1) *ϕ^G is irreducible,*
(2) *For each $x \in G - C_n$, there exists $y \in \langle a \rangle \cap x \langle a \rangle x^{-1}$ such that $xyx^{-1} \neq y$.*

DEFINITION. When the condition (2) of Proposition 3 holds, we say that G satisfies (EX, C).

Let H be a group. For a normal subgroup N of H, and any $g, h \in H$, we write
$g \equiv h \pmod{N}$,
when $g^{-1}h \in N$. For an element $g \in H$, we denote by $|g|$ the order of g.

4. Proof of Theorem

Let $\phi \in \mathrm{FIrr}(C_n)$. Since $\phi^G = (\phi^{N_1})^G \in \mathrm{Irr}(G)$, we must have $\phi^{N_1}\in \mathrm{Irr}(N_1)$. Therefore, by Theorem 0.1, we can take an element $b\in N_1-C_n=\{g\in N_1 \mid g\notin C_n\}$ such that

$$N_1=\langle a, b \mid a^{p^n}=b^{p^m}=1, \quad bab^{-1}=a^{1+p^{n-m}}\rangle\cong G(n, m).$$

for some $m\in \mathbf{N}$.

Note that any element in $N_1=\langle a, b\rangle$ is represented as a^ib^j for some $i, j\in \mathbf{Z}$, $0\leqq i\leqq p^n-1$, $0\leqq j\leqq p^m-1$.

To prove the theorem, we need the following:

Lemma 1. *Let p be an odd prime and n, m, k, j be integers satisfying $0\leqq m\leqq n-1$. Then, if we put $s=1+kp^{n-m}$, we have the following equality*:

$$\frac{s^{jp^m}-1}{s^j-1}\equiv p^m \pmod{p^n}.$$

Proof of Lemma 1. We can show this by direct calculations.

Lemma 2 *For any integers i, j, the following equalities hold.*

(i) $ab\equiv ba \pmod{\langle a^{p^{n-m}}\rangle}$.

(ii) $ba^{p^m}b^{-1}=a^{p^m}$.

(iii) $(a^ib^j)^{p^m}=a^{ip^m}$.

Proof of Lemma 2. (i), and (ii) can be shown by direct calculations.

(iii). If we put $s=1+p^{n-m}$, we have

$$(a^iu^j)^{p^m}=a^{i(s^{p^mj}-1/s^j-1)}u^{p^mj}=a^{p^mi},$$

by direct calculations and Lemma 1.

Let $f: N_2\to N_2/N_1$ be a natural epimorphism of groups. For $u\in N_2$, we write $o(u)$ for the order of $f(u)$ in N_2/N_1. We can show the following:

Claim I. If $u\in N_2$, then $o(u)\leqq p^m$.

Proof of Claim I. Take an element $x\in N_2$. and write $o(x)=p^t$. We will show that $t\leqq m$.

Write $xax^{-1}=a^{i_0}b^{j_0}$ and $xbx^{-1}=a^{d_0}b^{t_0}$. Since

$$xa^{p^m}x^{-1}=a^{p^mi_0},$$

by Lemma 2 (iii), we must have $(p, i_0)=1$.

On the other hand, since

$$1=xb^{p^m}x^{-1}=a^{d_0p^m},$$

we have

$$d_0\equiv 0 \pmod{p^{n-m}}.$$

Therefore, we may write $d_0=p^{n-m}d$ and

$$xbx^{-1}=a^{p^{n-m}d}b^{t_0},$$

for some $d\in \mathbf{Z}$. Since $n-m\geq m$, by our assumption, we have

$$xa^{p^{n-m}}x^{-1}=a^{p^{n-m}i_0}. \tag{1}$$

Taking the conjugate of both sides of the equality, $bab^{-1}=a^{1+p^{n-m}}$ by x, we get

$$(a^{p^{n-m}d}b^{t_0})(a^{i_0}b^{j_0})(a^{p^{n-m}d}b^{t_0})^{-1}=a^{i_0}b^{j_0}a^{p^{n-m}i_0}.$$

Hence, we have

$$a^{i_0(1+p^{n-m})^{t_0}}b^{j_0}=a^{i_0(1+p^{n-m})}b^{j_0}.$$

Therefore,

$$i_0(1+t_0\cdot p^{n-m})\equiv i_0(1+p^{n-m}) \pmod{p^n}.$$

But $(i_0, p)=1$, so we get $t_0\equiv 1 \pmod{p^m}$, and hence

$$xbx^{-1}=a^{p^{n-m}d}b. \tag{2}$$

Note that $\langle a^{p^{n-m}}\rangle$ is a normal subgroup of N_2, by (1). It is easy to see that

$$xbx^{-1}\equiv b \pmod{\langle a^{p^{n-m}}\rangle}.$$

$$ba\equiv ab \pmod{\langle a^{p^{n-m}}\rangle}.$$

Further, we have

$$xa^lx^{-1}=(a^{i_0}b^{j_0})^l\equiv a^{i_0l}b^{j_0l} \pmod{a^{p^{n-m}}\rangle}.$$

for any $l\in \mathbf{N}$.

Using these relations repeatedly, we get

$$x^sax^{-s}\equiv a^{i_0^s}b^{j_0(i_0^{s-1}+\cdot\cdot+i_0+1)} \pmod{\langle a^{p^{n-m}}\rangle},$$

for any $s\in \mathbf{N}$.

In particular,

$$x^{p^t}ax^{-p^t}\equiv a^{i_0^{p^t}}b^{j_0(i_0^{p^t-1}+\cdot\cdot+i_0+1)} \pmod{\langle a^{p^{n-m}}\rangle}.$$

Hence we may write as

$$x^{p^t}ax^{-p^t}=a^{i_0^{p^t}+rp^{n-m}}b^{j_0(i_0^{p^t-1}+\cdot\cdot+i_0+1)},$$

for some integer r.

Since $x^{p^t}\in N_1=\langle a, b\rangle$, we must have

$$i_0^{p^t}\equiv 1 \pmod{p^{n-m}}, \tag{3}$$

and

$$j_0(i_0^{p^t-1}+\cdot\cdot+i_0+1)\equiv 0 \pmod{p^m}. \tag{4}$$

By (2), we can write as $i_0=1+k_0p^s$, for some integers k_0 and s, $1\leqq s$. So,

$$j_0(i_0^{p^t-1}+\cdot\cdot+i_0+1)=j_0\left(\frac{i_0^{p^t}-1}{i_0-1}\right)=j_0p^tl,$$

for some integer l, $(p, l)=1$.

Suppose that $t\geqq m+1$, then

$$j_0(i_0^{p^{t-1}-1}+\cdot\cdot+i_0+1)\equiv 0 \pmod{p^m},$$

so, we have $x^{p^{t-1}}\in N_1$, which contradicts our hypothesis that $o(x)=p^t$. Thus the proof of Claim I is completed.

Take an element $x\in N_2$. Let $o(x)=p^t$, $0\leqq t\leqq m$, and write $xax^{-1}=a^{i_0}b^{j_0}$. We have shown, in the proof of Claim I, that

$$i_0^{p^t}\equiv 1 \pmod{p^{n-m}},$$

and

$$j_0p^t\equiv 0 \pmod{p^m},$$

So, we can write as $i_0=1+kp^{n-m-t}$, and $j_0=jp^{m-t}$ for some integers k and j, $(p, j)=1$.

Summarizing the results we can write as

$$xax^{-1}=a^{1+kp^{n-m-t}}b^{p^{m-t}j},$$

$$xbx^{-1}=a^{p^{n-m}d}b, \tag{5}$$

for some $k, j, d\in \mathbf{Z}$.

Define $t_0=max\{o(x) \mid x\in N_2\}$, and take an element $x_0\in N_2$ such that $o(x_0)=p^{t_0}$.

Denote by N_1^0 the subgroup of G generated by x_0 and the elements of N_1. Then

$$[N_1^0 : N_1]=p^{t_0} \text{ and } N_1^0/N_1\cong C_{t_0}.$$

We will show that

Claim II. $N_1^0=N_2$.

Proof of Claim II. Suppose that $N_1^0\subsetneqq N_2$. Take an element $w\in N_2-N_1^0=\{g\in N_2 \mid g\notin N_1^0\}$. Suppose that $o(y)=p^s$, then, by the hypothesis, we have $s\leqq t_0$. By the same way as in the proof of (5), we can write as

$$yay^{-1}=a^{1+k_1p^{n-m-s}}b^{p^{m-s}j_1},$$
$$yby^{-1}=a^{p^{n-m}d_1}b,$$

for some $k_1, j_1, d_1 \in \mathbf{Z}$, $(p, j_1)=1$.
On the other hand, we can take the element $u \in \langle x_0^{p^{t_0-s}} \rangle$, such that

$$u^{-1}au=a^{1+k_2p^{n-m-s}}b^{p^{m-s}(p^s-j_1)},$$
$$u^{-1}bu=a^{p^{n-m}d_2}b,$$

for some $k_2, d_2 \in \mathbf{Z}$.
We then have

$$u^{-1}yay^{-1}u=a^{1+k_3p^{n-m-s}},$$

for some $k_3 \in \mathbf{Z}$.

This means that $u^{-1}y \in N_1$. So, we have $y \in N_1^0$, which contradicts our hypothesis. Thus the proof of Claim II is completed.

By Claim II, we have $N_2/N_1 = N_1^0/N_1 \cong C_{t_0}$, $t_0 \leqq m$. So the proof of the theorem is completed.

References

[1] C. Curtis and I. Reiner: ''Representation theory of finite groups and associative algebras'', Interscience, New York, 1962.

[2] Y. Iida: *The p-groups with an irreducible character induced from a faithful linear character*, Preprint.

[3] Y. Iida and T. Yamada: *Extensions and induced characters of quaternion, dihedral and semidihedral groups*, SUT J. Math. 27 (1991), 237–262.

[4] Y. Iida and T. Yamada: *Types of faithful metacyclic* 2-*groups*, SUT J. Math. 28 (1992), 23–46.

[5] K. Sekiguchi: *Extensions of some* 2-*groups which preserve the irreducibilities of induced characters*, Osaka J. Math. 37 (2000).

[6] K. Sekiguchi: *Irreducibilities of the induced characters of cyclic p-groups*, Math. J. of Okayama Univ. 41 (1999).

[7] K. Sekiguchi: *Extensions and the irreducibilities of the induced characters of cyclic p-groups*, Hiroshima Math. J. 32 (2002).

[8] K. Sekiguchi: *Extensions and the irreducibilities of induced characters of some* 2-*groups*, Hokkaido Math. J. 31 (2002).

[9] H. Zassenhaus: ''The theory of groups'', Chelsea, New York, 1949.