

The strength of graphs and related invariants

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Abstract: A numbering f of a graph G of order n is a labeling that assigns distinct elements of the set $\{1, 2, \dots, n\}$ to the vertices of G . The strength of G is

$$\text{str}(G) = \min \{ \text{str}_f(G) \mid f \text{ is a numbering of } G \},$$

where $\text{str}_f(G) = \max \{ f(u) + f(v) \mid uv \in E(G) \}$. In this paper, we introduce the concept of anti-strength $\text{astr}(G)$, and establish that $\text{str}(G) + \text{astr}(G) = 2(n+1)$ for a nonempty graph G of order n . In addition, we show how the strength (or anti-strength) of a graph and other invariants defined on graphs are related.

Key Words: strength, anti-strength, bandwidth, cartesian product, join operation, corona operation

1. Introduction

In this paper, we will consider only finite graphs without loops or multiple edges. We refer the reader to the book by Chartrand and Lesniak [1] for graph-theoretical notation and terminology not described in this paper. The *path*, the *cycle* and the *complete graph* of order n are denoted by P_n , C_n and K_n , respectively. The *complete bipartite graph* with partite sets V_1 and V_2 , where $|V_1| = s$ and $|V_2| = t$, is denoted by $K_{s,t}$. The graph with n vertices and no edges is referred to as the *empty graph*.

The *degree of a vertex* v in a graph is number of edges of G incident with v and is denoted by $\deg_G v$. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. The *maximum degree* is defined similarly and is denoted by $\Delta(G)$.

The *cartesian product* $G = G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. Some important classes of graphs can be defined in terms of cartesian products. The *ladder* L_n can be defined as the graph $P_n \times K_2$. The *prism* D_n can be defined as the graph $C_n \times K_2$. The *hypercube* Q_n can be defined inductively as $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$ for any integer $n \geq 2$.

Among all graph labeling problems, bandwidth numbering of graphs has perhaps attracted the most attention in the literature. The bandwidth numbering was independently proposed by Harary [8] and Harper [9]. The motivation for these numberings came from the study of sparse matrix computations, representing data structures by linear arrays, VLSI layouts and mutual simulations of interconnection networks (see [2, 3, 12]).

For the sake of notational convenience, we will denote the interval of integers k such that $i \leq k \leq j$ by

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simply writing $[i, j]$. A *numbering* f of a graph G of order n is a labeling that assigns distinct elements of the set $[1, n]$ to the vertices of G . The *bandwidth* of G is

$$\text{band}(G) = \min \{ \text{band}_f(G) \mid f \text{ is a numbering of } G \},$$

where $\text{band}_f(G) = \max \{ |f(u) - f(v)| \mid uv \in E(G) \}$. An additive analogous for bandwidth numberings of graphs has been introduced in [10]. The *strength* of G is

$$\text{str}(G) = \min \{ \text{str}_f(G) \mid f \text{ is a numbering of } G \},$$

where $\text{str}_f(G) = \max \{ f(u) + f(v) \mid uv \in E(G) \}$.

Several sharp bounds for the strength of a graph have been found in terms of other invariants defined on graphs (see [10]). Among others, the following result that provides a lower bound for the strength of a graph in terms of its order and minimum degree will prove to be useful in this paper.

Lemma 1.1. *For every graph G of order n with $\delta(G) \geq 1$,*

$$\text{str}(G) \geq n + \delta(G).$$

It is worth to mention that all the class of graphs whose strength have been determined thus far attain the lower bound given in Lemma 1.1 (see Table 1, which summarizes what has been known about such graphs). In this table, the *Möbius ladder* (a graph obtained from the ladder $P_n \times K_2$ by joining the opposite end-vertices of the two copies of P_n) is denoted by M_n . The *k-level complete n-ary tree* (a tree in which the i th level consists of n^{i-1} vertices and each vertex in level $i < k$ has n ‘sons’ at level $i + 1$) is denoted by $T_{n,k}$. If $n = 2$, then $T_{n,k}$ is referred to as a *k-level complete binary tree*. A *caterpillar* (introduced by Harary and Schwenk [6]) is a tree T with the property that the removal of the end-vertices of T results in a path.

Table 1. Summary of strengths of graphs

G	$ V(G) $	$\delta(G)$	$\text{str}(G)$
P_n ($n \geq 2$)	n	1	$n + 1$ [10]
K_n ($n \geq 2$)	n	$n - 1$	$2n - 1$ [10]
C_n ($n \geq 3$)	n	2	$n + 2$ [10]
$K_{s,t}$ ($s \in [1, t]$)	$s + t$	s	$2s + t$ [10]
nP_2 ($n \geq 1$)	$2n$	1	$2n + 1$ [10]
L_n ($n \geq 2$)	$2n$	2	$2n + 2$ [10]
M_n ($n \geq 2$)	$2n$	3	$2n + 3$ [10]
D_n ($n \geq 3$)	$2n$	3	$2n + 3$ [10]
$K_{s,t} \times K_2$ ($s \in [1, t]$)	$2(s + t)$	$s + 1$	$3s + 2t + 1$ [10]
Q_n ($n \in [1, 3]$)	2^n	n	$2^n + n$ [10]
$T_{n,k}$ ($n, k \geq 2$)	$(n^k - 1) / (n - 1)$	1	$(n^k - 1) / (n - 1) + 1$ [11]
caterpillars	n	1	$n + 1$ [11]

For the join of two graphs, the next lower and upper bounds were found in [10].

Theorem 1.1. *If G is a nonempty graph of order n , then*

$$n + m + \min \{ n, \delta(G) + m \} \leq \text{str}(G + mK_1) \leq \text{str}(G) + 2m$$

for every positive integer m .

The *join* $G \cong G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

Using the join operation, the *wheel* W_n can be defined as the graph $C_n + K_1$. Since $\delta(C_n) = 2$ and $\text{str}(C_n) = n + 2$ (see Table 1), it follows from Theorem 1.1 that $\text{str}(W_n) = n + 4$ for integers $n \geq 3$.

If G_1 has order p , the *corona* $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p copies of G_2 and joining the i th vertex of G_1 with an edge to every vertex in the i th copy of G_2 . For the corona of two graphs, the following result was established in [11].

Theorem 1.2. *If G is a graph of order n with $\delta(G) \geq 1$ and $\text{str}(G) = n + \delta(G)$, then*

$$\text{str}(G \odot mK_1) = (m + 1)n + 1$$

for every positive integer m .

As we have seen in Table 1, there are infinitely many graphs G for which $\delta(G) \geq 1$ and $\text{str}(G) = |V(G)| + \delta(G)$. Therefore, applying Theorem 1.2 with these classes of graphs repeatedly, we readily obtain other classes of graphs H satisfying $\delta(H) \geq 1$ and $\text{str}(H) = |V(H)| + \delta(H)$.

In this paper, we introduce a dual concept of the strength called the anti-strength $\text{astr}(G)$, and establish that $\text{str}(G) + \text{astr}(G) = 2(n + 1)$ for a nonempty graph G of order n . In addition, we show how the strength (or anti-strength) of a graph and other invariants defined on graphs are related.

2. The anti-strength of graphs

For a numbering f of a graph G of order n , the *complementary numbering* \bar{f} of G is defined by

$$\bar{f}(v) = n + 1 - f(v)$$

for each $v \in V(G)$. Then $\text{band}_{\bar{f}}(G) = \text{band}_f(G)$. Thus, the complementary numbering of any bandwidth numbering is also a bandwidth numbering. However, this is not true for strength numberings as the subsequent comment indicates.

Notice now that the numbering of a graph G of order n and its complementary numbering satisfies

$$\min \{\bar{f}(u) + \bar{f}(v) \mid uv \in E(G)\} = 2(n + 1) - \max \{f(u) + f(v) \mid uv \in E(G)\}$$

and

$$\max \{\bar{f}(u) + \bar{f}(v) \mid uv \in E(G)\} = 2(n + 1) - \min \{f(u) + f(v) \mid uv \in E(G)\}.$$

This naturally suggests us the following problem that is a dual modification of the strength problem. The *anti-strength* $\text{astr}(G)$ of a graph G of order n is

$$\text{astr}(G) = \max \{\text{astr}_f(G) \mid f \text{ is a numbering of } G\},$$

where $\text{astr}_f(G) = \min \{f(u) + f(v) \mid uv \in E(G)\}$. The following result indicates how the strength of a nonempty graph and its anti-strength are related.

Lemma 2.1. *For every nonempty graph G of order n ,*

$$\text{str}(G) + \text{astr}(G) = 2(n + 1).$$

Proof. Let G be a nonempty graph of order n and $\text{str}(G) = k$. Then there exists a numbering $f : V(G) \rightarrow [1, n]$ of G for which

$$\text{str}(G) = \min \{ \text{str}_f(G) \mid f \text{ is a numbering of } G \} = k,$$

that is, $\text{str}_f(G) = \max \{ f(u) + f(v) \mid uv \in E(G) \} = k$. Thus, the complementary numbering \bar{f} of G has the property that

$$\begin{aligned} \text{astr}_{\bar{f}}(G) &= \min \{ \bar{f}(u) + \bar{f}(v) \mid uv \in E(G) \} \\ &= 2(n+1) - \max \{ f(u) + f(v) \mid uv \in E(G) \} \\ &= 2(n+1) - k. \end{aligned}$$

Consequently, $\text{astr}(G) \geq 2(n+1) - k$, implying that $\text{str}(G) + \text{astr}(G) \geq 2(n+1)$.

To show that $\text{str}(G) + \text{astr}(G) \leq 2(n+1)$, let $\text{astr}(G) = k$. Then there exists a numbering $f : V(G) \rightarrow [1, n]$ of G for which

$$\text{astr}(G) = \max \{ \text{astr}_f(G) \mid f \text{ is a numbering of } G \} = k,$$

that is, $\text{astr}_f(G) = \min \{ f(u) + f(v) \mid uv \in E(G) \} = k$. Thus, the complementary numbering \bar{f} of G has the property that

$$\begin{aligned} \text{str}_{\bar{f}}(G) &= \max \{ \bar{f}(u) + \bar{f}(v) \mid uv \in E(G) \} \\ &= 2(n+1) - \min \{ f(u) + f(v) \mid uv \in E(G) \} \\ &= 2(n+1) - k. \end{aligned}$$

Consequently, $\text{str}(G) \leq 2(n+1) - k$, implying that $\text{str}(G) + \text{astr}(G) \leq 2(n+1)$ and the proof is completed. \square

From the above result, it follows that the problems of determining the strength and anti-strength are equivalent for nonempty graphs. Therefore, applying the same result with the classes of graphs G contained in Table 1, we can obtain the formulas for $\text{astr}(G)$ as summarized in Table 2. Moreover, the bounds for $\text{str}(G)$ found in [10] together with Lemma 2.1 provide the parallel bounds for $\text{astr}(G)$.

Table 2. Summary of anti-strengths of graphs

G	$ V(G) $	$\text{astr}(G)$
P_n ($n \geq 2$)	n	$n+1$
K_n ($n \geq 2$)	n	3
C_n ($n \geq 3$)	n	n
$K_{s,t}$ ($s \in [1, t]$)	$s+t$	$t+2$
nP_2 ($n \geq 1$)	$2n$	$2n+1$
L_n ($n \geq 2$)	$2n$	$2n$
M_n ($n \geq 2$)	$2n$	$2n-1$
D_n ($n \geq 3$)	$2n$	$2n-1$
$K_{s,t} \times K_2$ ($s \in [1, t]$)	$2(s+t)$	$s+2t+1$
Q_n ($n \in [1, 3]$)	2^n	$2^n - n + 2$
$T_{n,k}$ ($n, k \geq 2$)	$(n^k - 1) / (n - 1)$	$(n^k - 1) / (n - 1) + 1$
caterpillars	n	$n+1$

3. Relations Among Invariants

In this section, we discuss briefly how the strength (anti-strength) of a graph and other invariants in graph theory are related.

The decision problem associated with determining the bandwidth of an arbitrary graph was shown to be NP-complete by Papadimitriou [13]. Garey et al. [7] proved that the problem remains to be NP-complete for trees with maximum degree 3. Hence, a direction of research is to find sharp bounds for the bandwidth of a graph. Several sharp bounds for the bandwidth of a graph have been found in terms of other invariants defined on graphs. We first state one of them listed in [2].

Lemma 3.1. *For every graph G ,*

$$\text{band}(G) \geq \delta(G).$$

The next result provides a lower bound for the strength of certain graphs in terms of its order and bandwidth. The proof follows at once from Lemma 1.1.

Corollary 3.1. *For every graph G of order n with $\text{band}(G) = \delta(G) \geq 1$,*

$$\text{str}(G) \geq n + \text{band}(G).$$

The next result is immediately obtained from Lemma 3.1.

Corollary 3.2. *For every nonempty graph G of order n with $\text{str}(G) = n + \delta(G)$,*

$$\text{str}(G) \leq n + \text{band}(G).$$

It is known from [14] that

$$\begin{aligned} \text{band}(P_n) &= 1 \ (n \geq 2), & \text{band}(K_n) &= n - 1 \ (n \geq 2), \\ \text{band}(C_n) &= 2 \ (n \geq 3), & \text{band}(L_n) &= 2 \ (n \geq 2). \end{aligned}$$

Hence, these classes of graphs G satisfy $\text{band}(G) = \delta(G)$. From Table 1, these classes of graphs G also attain the bound presented in Corollary 3.1. Moreover, it is known from [14] that

$$\begin{aligned} \text{band}(M_n) &= 4 \ (n \geq 2), & \text{band}(D_n) &= 4 \ (n \geq 3), \\ \text{band}(W_n) &= \lfloor n/2 \rfloor \ (n \geq 6), & \text{band}(K_{s,t}) &= \lceil t/2 \rceil + s + 1 \ (s \in [1, t]). \end{aligned}$$

This together with Table 1 and the aforementioned fact that $\text{str}(W_n) = n + 4 \ (n \geq 3)$ (see the comment subsequent to Theorem 1.1) illustrates that the inequality given in Corollary 3.2 holds for these classes of graphs.

The next two results follow readily from the preceding results and Lemma 2.1.

Corollary 3.3. *For every nonempty graph G of order n with $\text{band}(G) = \delta(G)$,*

$$\text{astr}(G) \leq n + 2 - \text{band}(G).$$

Corollary 3.4. *For every nonempty graph G of order n with $\text{str}(G) = n + \delta(G)$,*

$$\text{astr}(G) \geq n + 2 - \text{band}(G).$$

For a connected graph G and a pair $u, v \in V(G)$, the *distance* $d_G(u, v)$ between u and v is the length of a shortest $u - v$ path in G . The *diameter* $\text{diam}G$ of a connected graph G is defined as $\text{diam}G = \max \{d_G(u, v) \mid u, v \in V(G)\}$.

The following result provides lower and upper bounds for the bandwidth of a connected graph in terms of its order and diameter. The lower bound is due to Chvátal [4] and the upper bound to Chvátalová et al. [5].

Lemma 3.2. *For every connected graph G of order n ,*

$$\left\lceil \frac{n-1}{\text{diam}G} \right\rceil \leq \text{band}(G) \leq n - \text{diam}G.$$

The next result provides lower and upper bounds for the strength of a connected graph in terms of its order and diameter.

Lemma 3.3. *For every connected graph G of order n ,*

$$\left\lceil \frac{2n-1}{\text{diam}G} \right\rceil \leq \text{str}(G) \leq 2n - \text{diam}G.$$

Proof. The upper bound has been shown in [10]. To verify the lower bound, let $P : v_0, v_1, \dots, v_k$ be a $u - v$ path of length k satisfying $f(u) = n$ and $f(v) = n - 1$ for any numbering $f : V(G) \rightarrow [1, n]$, where $u = v_0$ and $v = v_k$. Then

$$\begin{aligned} \text{str}_f(G) &\geq \max \{f(x) + f(y) \mid xy \in E(P)\} \\ &\geq \frac{1}{k} \sum_{i=0}^{k-1} (f(v_i) + f(v_{i+1})) \\ &\geq \frac{1}{\text{diam}G} \sum_{i=0}^{k-1} (f(v_i) + f(v_{i+1})) \\ &\geq \frac{f(u) + f(v)}{\text{diam}G} = \frac{2n-1}{\text{diam}G}, \end{aligned}$$

which yields the desired result. □

The bounds presented in Lemma 3.3 are always attainable. Simply, let $G = K_n$. Then $\text{diam}G = 1$ and $\text{str}(G) = 2n - 1$ for any integer $n \geq 2$ (see Table 1).

The lower bound given in the preceding lemma can be extended to obtain the following result.

Theorem 3.1. *For every connected graph G ,*

$$\text{str}(G) \geq \max_k \max \left\{ \frac{f(u) + f(v)}{\text{diam}S} \mid u, v \in S \subseteq V(G), \text{diam}S = k \right\},$$

where $\text{diam}S = \max \{d_G(x, y) \mid x, y \in S\}$.

Proof. For any numbering $f : V(G) \rightarrow [1, |V(G)|]$, select two vertices u and v of G such that

$$f(u) = \max \{f(x) \mid x \in S\} \text{ and } f(v) = \max \{f(x) \mid x \in S - \{u\}\},$$

which gives the desired result. □

The next two results now follow from the preceding results and Lemma 2.1.

Corollary 3.5. *For every connected graph G of order n ,*

$$2 + \text{diam}G \leq \text{astr}(G) \leq 2(n+1) - \left\lceil \frac{2n-1}{\text{diam}G} \right\rceil.$$

Corollary 3.6. *For every connected graph G of order n ,*

$$\text{astr}(G) \leq 2(n+1) - \max_k \max \left\{ \frac{f(u) + f(v)}{\text{diam}S} \mid u, v \in S \subseteq V(G), \text{diam}S = k \right\},$$

where $\text{diam}S = \max \{d_G(x, y) \mid x, y \in S\}$.

The next result provides a formula for the strength of a connected graph in terms of its order and bandwidth.

Theorem 3.2. *Let G be a connected graph of order n for which $n = \delta(G) + \text{diam}G$. Then*

$$\text{str}(G) = n + \text{band}(G).$$

Proof. Let G be a connected graph of order n , and assume that $n = \delta(G) + \text{diam}G$. By Lemma 3.2, $\delta(G) \geq \text{band}(G)$, and hence it follows from Lemma 1.1 that

$$\text{str}(G) \geq n + \delta(G) \geq n + \text{band}(G).$$

By Lemma 3.3, $\text{str}(G) \leq n + \delta(G)$, and hence it follows from Lemma 3.1 that

$$\text{str}(G) \leq n + \delta(G) \leq n + \text{band}(G).$$

This produces the desired result. □

The proof of Theorem 3.2 provides another result.

Corollary 3.7. *Let G be a connected graph of order n for which $n = \delta(G) + \text{diam}G$. Then*

- (1) $\text{band}(G) = \delta(G)$,
- (2) $\text{band}(G) = n - \text{diam}G$,
- (3) $\text{str}(G) = n + \delta(G)$,
- (4) $\text{str}(G) = 2n - \text{diam}G$.

This result also has a rather immediate corollary.

Corollary 3.8. *Let G be a connected graph of order n for which $n = \delta(G) + \text{diam}G$. Then*

- (1) $\text{astr}(G) = n + 2 - \text{band}(G)$,
- (2) $\text{astr}(G) = n + 2 - \delta(G)$,
- (3) $\text{astr}(G) = 2 + \text{diam}G$.

No general formula for the bandwidth, strength and anti-strength of graphs is known. Indeed, it is unlikely that such a formula will ever be developed in terms of quantities that are easily computable. However, there are infinitely many connected graphs G such that $|V(G)| = \delta(G) + \text{diam}G$, for instance, $G = P_n, K_n$ ($n \geq 2$). This motivates us to propose the next problem.

Problem 1. Find sufficient conditions for a connected graph G of order n to satisfy $n = \delta(G) + \text{diam}G$.

Lemma 3.3 gives rise to another formula for the strength of a connected graph.

Theorem 3.3. Let G be a connected graph of order n for which $n = \Delta(G) + \text{diam}G$, $\text{band}(G) = \Delta(G)$ and $\text{str}(G) \geq n + \text{band}(G)$. Then

$$\text{str}(G) = n + \text{band}(G) = n + \Delta(G).$$

Proof. Consider such a graph G . Then it follows from Lemma 3.3 that

$$\begin{aligned}\text{str}(G) &\leq 2n - \text{diam}G \leq 2n - (n - \Delta(G)) \\ &= n + \Delta(G) = n + \text{band}(G).\end{aligned}$$

This provides the desired result. □

The following result is a consequence of Lemma 2.1 and Theorem 3.3.

Corollary 3.9. Let G be a connected graph of order n for which $n = \Delta(G) + \text{diam}G$, $\text{band}(G) = \Delta(G)$ and $\text{str}(G) \geq n + \text{band}(G)$. Then

$$\text{astr}(G) = n + 2 - \text{band}(G) = n + 2 - \Delta(G).$$

Notice that the complete K_n ($n \geq 2$) satisfies the hypothesis of Theorem 3.3. This leads us to propose the next problem.

Problem 2. Find sufficient conditions for a connected graph G of order n to satisfy $n = \Delta(G) + \text{diam}G$, $\text{band}(G) = \Delta(G)$ and $\text{str}(G) \geq n + \text{band}(G)$.

References

- [1] G. Chartrand and L. Lesniak, *Graphs & Digraphs* 3th ed. CRC Press (1996).
- [2] P.Z. Chinn, J. Chvátalová, A.K. Dewdney, and N.E. Gibbs, The bandwidth problem for graphs and matrices – a survey, *J. Graph Theory*, **6** (1982) 223–254.
- [3] F.R.K. Chung, Labelings of graphs, in L.W. Beineke and R.J. Wilson (ed.), *Selected Topics in Graph Theory*, Vol. 3 (1988), 151–168.
- [4] V. Chvátal, A remark on a problem of Harary, *Czech. Math. J.*, **20** (1970) 109–111.
- [5] J. Chvátalová, A.K. Dewdney, N.E. Gibbs and R.R. Korfhage, The bandwidth problem for graphs: a collection of recent results. Research Report #24, Department of Computer Science, UWO, London, Ontario (1975).
- [6] F. Harary and A.J. Schwenk, The number of caterpillars, *Discrete Math.*, **6** (1973) 359–365.
- [7] M.R. Garey, R.L. Graham, D.S. Johnson and D.E. Knuth, Complexity results for bandwidth minimization, *SIAM J. Appl. Math.*, **34** (1978) 477–495.
- [8] F. Harary, Problem 16, In: *Theory of Graphs and its Applications* (ed. M. Fiedler), Czech. Acad. Sci., Prague (1967), 161.
- [9] L.H. Harper, Optimal numberings and isoperimetric problems on graphs, *J. Combin. Theory*, **1** (1966) 385–393.
- [10] R. Ichishima, F.A. Muntaner-Batlé, and A. Oshima, Bounds for the strength of graphs, *Australas. J. Combin.*, **72** (3) (2018) 492–508.
- [11] R. Ichishima, F.A. Muntaner-Batlé, and A. Oshima, On the strength of some trees, *AKCE Int. J. Graphs Comb.*, (2019), <https://doi.org/10.1016/j.akcej.2019.06.002>.
- [12] Y.L. Lai and K. Williams, A survey of solved problems and applications on bandwidth, edgesum, and profile of graphs, *J. Graph Theory*, **31** (2) (1999) 75–94.
- [13] C.H. Papadimitriou, The NP-completeness of the bandwidth minimization problem, *Computing*, **16** (1976) 263–270.
- [14] E. Weisstein, Graph bandwidth, <http://mathworld.wolfram.com/GraphBandwidth.html> (2020). Accessed 29 January 2020.