Paper

The Edge Dependent Characteristic of Cycles

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Abstract: A graph G is called strongly k-indexable if there exists a bijective function $f : V(G) \rightarrow \{0, 1, \ldots, |V(G)| - 1\}$ such that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting edge labels is $\{k, k+1, \ldots, k + |E(G)| - 1\}$ for some positive integer k. The edge dependent characteristic of a graph G is either the smallest nonnegative integer n with the property that $G \cup nK_2$ is strongly k-indexable for some positive integer k or $+\infty$ if there exists no such integer n. In this paper, we provide the formula for the edge dependent characteristic of all cycles, which settles the question raised by Hegde and Shetty [6].

1. Introduction

Only graphs without loops or multiple edges will be considered in this paper. Undefined graph theoretical notation and terminology can be found in [2]. The *vertex set* and *edge set* of a graph G are represented by $V(G)$ and $E(G)$, respectively. The *union* $G_1 \cup G_2$ of two subgraphs G_1 and G_2 of a graph G is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. As usual, the *cycle* with n vertices is denoted by C_n.

For the sake of notational convenience, we will denote the interval of integers k such that $i \le k \le j$ by simply writing $[i, j]$. On the other hand, if $a > b$, then we treat $[a, b]$ as the empty set. If such situations appear in particular formulas for a given vertex labeling, then we ignore the corresponding portions of formulas.

The notion of a strongly k-indexable graph was introduced in 1991 by Acharya and Hegde [1]. A graph G is called *strongly k-indexable* if there exists a bijective function $f : V(G) \rightarrow [0, |V(G)| - 1]$ such that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting edge labels is $[k, k + |E(G)| - 1]$ for some positive integer k.

The notion of edge-magic labelings was introduced in 1970 by Kotzig and Rosa [8]. These labelings were originally called "magic valuations" by them. These were rediscovered in 1996 by Ringel and Lladó [9] who coined one of the now popular terms for them: edge-magic labelings. For a graph G, a bijective function $f : V(G) \cup E(G) \rightarrow [1, |V(G)| + |E(G)|]$ is called an *edge-magic labeling* if $f(u) + f(v) + f(uv)$ is a constant (called the valence of f) for each $uv \in E(G)$. If such a labeling exists, then G is called an *edge-magic graph*. In 1998, Enomoto et al. [3] defined a slightly restricted version of an edge-magic labeling f of a graph G by requiring that $f(V(G)) = [1, |V(G)|]$. Such a labeling was called by them *super edge-magic*. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling.

The following result found in [4] provides us with a necessary and sufficient condition for a graph to be super edge-magic.

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Lemma 1. *A graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to [1, |V(G)|]$ such that the set*

$$\{f(u) + f(v) \,|\, uv \in E(G)\}$$

consists of $|E(G)|$ consecutive integers.

In light of Lemma 1, it is sufficient to exhibit the vertex labeling of a super edge-magic graph.

According to the latest version of dynamic survey by Gallian [5] available to authors, Hegde and Shetty [7] established the following relation among super edge-magic and strongly k-indexable graphs in 2002.

Lemma 2. *A graph G is super edge-magic with valence $val(G)$ if and only if G is strongly k-indexable, where $k = val(G) - |V(G)| - |E(G)|$.*

2. The Edge Dependent Characteristic of Cycles

The notion of edge dependent characteristic was introduced by Hegde and Shetty [6]. The *edge dependent characteristic $e_c(G)$* of a graph G is either the smallest nonnegative integer n with the property that $G \cup nK_2$ is strongly k-indexable for some positive integer k or $+\infty$ if there exists no such integer n. They further showed that $e_c(C_6) = 1$ and constructed polygon $\mathcal{P}_{12}$ of equal internal angles with sides of distinct lengths from a strongly 4-indexable labeling of $C_6 \cup K_2$. Moreover, they asked to compute $e_c(C_{4n+2})$ for every positive integer n. In this section, we answer this question by determining the formula for the edge dependent characteristic of all cycles.

The following result, due to Enomoto et al [3], characterizes super edge-magic cycles.

Theorem 1. *The cycle C_n is super edge-magic if and only if $n \geq 3$ is odd.*

With the aid of Theorem 1, it is now possible to provide the formula for the edge dependent characteristic of cycles as the next result indicates.

Theorem 2. *For every integer $n \geq 3$,*

$$e_c(C_n) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. First, assume that n is odd. In this case, Enomoto et al [3] found a super edge-magic labeling of C_n with valence $(5n + 3)/2$. It follows from Lemma 2 that C_n is strongly $(n+3)/2$-indexable, that is, $e_c(C_n) = 0$.

Next, assume that n is even. Then, by Theorem 1 and Lemma 2, the cycle C_n is not strongly k-indexable and thus $e_c(C_n) \geq 1$. To show that $e_c(C_n) \leq 1$, let $G \cong C_n \cup K_2$ be the graph with

$$V(G) = \{u_i | i \in [1, m]\} \cup \{x, y\}$$

and

$$E(G) = \{u_1 u_m\} \cup \{u_i u_{i+1} | i \in [1, m-1]\} \cup \{xy\},$$

and consider the following cases according to the possible values for the integer n.

Case 1: For $n = 4$, define the vertex labeling $f : V(G) \to [1, 6]$ such that

$$(f(u_i))_{i=1}^{4} = (2, 3, 5, 4);$$
$$f(x) = 1; \text{ and}$$
$$f(y) = 6.$$

Then, by Lemma 1, f extends to a super edge-magic labeling of G with valence 16. Thus, by Lemma 2, G is strongly 3-indexable.

Case 2: For $n = 6$, define the vertex labeling $f : V(G) \to [1, 8]$ such that

$$(f(u_i))_{i=1}^{6} = (2, 4, 3, 7, 5, 6);$$
$$f(x) = 1; \text{ and}$$
$$f(y) = 8.$$

Then, by Lemma 1, f extends to a super edge-magic labeling of G with valence 21. Thus, by Lemma 2, G is strongly 4-indexable.

Case 3: For $n = 10$, define the vertex labeling $f : V(G) \to [1, 12]$ such that

$$(f(u_i))_{i=1}^{10} = (2, 6, 3, 7, 4, 11, 5, 9, 8, 10);$$
$$f(x) = 1; \text{ and}$$
$$f(y) = 12.$$

Then, by Lemma 1, f extends to a super edge-magic labeling of G with valence 31. Thus, by Lemma 2, G is strongly 6-indexable.

Case 4: For $n = 14$, define the vertex labeling $f : V(G) \to [1, 16]$ such that

$$(f(u_i))_{i=1}^{14} = (2, 10, 3, 11, 4, 6, 5, 13, 9, 15, 8, 12, 7, 14);$$
$$f(x) = 1; \text{ and}$$
$$f(y) = 16.$$

Then, by Lemma 1, f extends to a super edge-magic labeling of G with valence 41. Thus, by Lemma 2, G is strongly 8-indexable.

Case 5: For $m = 8k$, where k is a positive integer, define the vertex labeling $f : V(G) \to [1, 8k + 2]$ such that

$$f(u_j) = \begin{cases} 4k + i + 1, & \text{if } j = 2i - 1 \text{ and } i \in [1, 2k]; \\ i, & \text{if } j = 2i \text{ and } i \in [1, 2k]; \\ 6k + 2i + 2, & \text{if } j = 4k + 4i - 3 \text{ and } i \in [1, k]; \\ 2k + 2i + 1, & \text{if } j = 4k + 4i - 2 \text{ and } i \in [1, k]; \\ 6k + 2i + 1, & \text{if } j = 4k + 4i - 1 \text{ and } i \in [1, k]; \\ 2k + 2i, & \text{if } j = 4k + 4i \text{ and } i \in [1, k]; \end{cases}$$

$f(x) = 2k + 1$; and $f(y) = 6k + 2$. Then, by Lemma 1, f extends to a super edge-magic labeling of G with valence $20k + 6$. Thus, by Lemma 2, G is strongly $(4k + 1)$-indexable.

Case 6: For $m = 8k + 4$, where k is a positive integer, define the vertex labeling $f : V(G) \to [1, 8k + 6]$

such that

$$
f(u_j) = \begin{cases}
4k + i + 3, & \text{if } j = 2i - 1 \text{ and } i \in [1, 2k + 1]; \\
i, & \text{if } j = 2i \text{ and } i \in [1, 2k + 1]; \\
6k + 2i + 5, & \text{if } j = 4k + 4i - 1 \text{ and } i \in [1, k]; \\
2k + 2i + 2, & \text{if } j = 4k + 4i \text{ and } i \in [1, k - 1]; \\
6k + 2i + 4, & \text{if } j = 4k + 4i + 1 \text{ and } i \in [1, k - 1]; \\
2k + 2i + 1, & \text{if } j = 4k + 4i + 2 \text{ and } i \in [1, k - 1]; \\
4k + 1, & \text{if } j = 8k; \\
8k + 4, & \text{if } j = 8k + 1; \\
4k + 3, & \text{if } j = 8k + 2; \\
8k + 6, & \text{if } j = 8k + 3; \\
4k + 2, & \text{if } j = 8k + 4;
\end{cases}
$$

$f(x) = 2k + 2$; and $f(y) = 6k + 5$. Then, by Lemma 1, f extends to a super edge-magic labeling of G with valence $20k + 16$. Thus, by Lemma 2, G is strongly $(4k + 3)$-indexable.

Case 7: For $m = 8k + 10$, where k is a positive integer, define the vertex labeling $f : V(G) \to [1, 8k + 12]$ such that

$$
f(u_j) = \begin{cases}
i + 1, & \text{if } j = 2i - 1 \text{ and } i \in [1, 2k + 4]; \\
4k + i + 5, & \text{if } j = 2i \text{ and } i \in [1, 2k + 2]; \\
6k + 11, & \text{if } j = 4k + 6; \\
6k + 9, & \text{if } j = 4k + 8; \\
6k + 8, & \text{if } j = 4k + 9; \\
6k + 2i + 8, & \text{if } j = 4k + 4i + 6 \text{ and } i \in [1, k + 1]; \\
2k + 2i + 5, & \text{if } j = 4k + 4i + 7 \text{ and } i \in [1, k]; \\
6k + 2i + 11, & \text{if } j = 4k + 4i + 8 \text{ and } i \in [1, k]; \\
2k + 2i + 4, & \text{if } j = 4k + 4i + 9 \text{ and } i \in [1, k];
\end{cases}
$$

$f(x) = 1$; and $f(y) = 8k + 12$. Then, by Lemma 1, f extends to a super edge-magic labeling of G with valence $20k + 31$. Thus, by Lemma 2, G is strongly $(4k + 6)$-indexable.

Case 8: For $m = 8k + 14$, where k is a positive integer, define the vertex labeling $f : V(G) \to [1, 8k + 16]$ such that

$$
f(u_j) = \begin{cases}
i + 1, & \text{if } j = 2i - 1 \text{ and } i \in [1, 2k + 4]; \\
4k + i + 9, & \text{if } j = 2i \text{ and } i \in [1, k + 2]; \\
5k + i + 12, & \text{if } j = 2k + 2i + 4 \text{ and } i \in [1, k]; \\
2k + 6, & \text{if } j = 4k + 6; \\
6k + 13, & \text{if } j = 4k + 8; \\
4k - i + 10, & \text{if } j = 4k + 2i + 7 \text{ and } i \in [1, 2k + 3]; \\
8k - i + 16, & \text{if } j = 4k + 2i + 8 \text{ and } i \in [1, k + 1]; \\
5k + 12, & \text{if } j = 6k + 12; \\
7k - i + 15, & \text{if } j = 6k + 2i + 12 \text{ and } i \in [1, k + 1];
\end{cases}
$$

$f(x) = 1$; and $f(y) = 8k + 16$. Then, by Lemma 1, f extends to a super edge-magic labeling of G with valence $20k + 41$. Thus, by Lemma 2, G is strongly $(4k + 8)$-indexable.

Therefore, we conclude that $e_c(G) = 1$ when n is even, completing the proof. $\qquad\square$

Acknowledgments

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