論文　**Original Paper**

Linear canonical transformations and transformation functions of the squeeze operator

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Abstract: We investigate the transformation functions of a general time-dependent linear transformation in coordinate-momentum phase space using the "Integration Within Ordered Product" (IWOP) technique. These transformation functions comprise a classical generating function which engenders a linear canonical transformation in coordinate-momentum phase space. This reveals a new correspondence between classical and quantum mechanics.

Key words: Quantum mechanics, Canonical transformation, Hamilton-Jacobi equation

Ⅰ. INTRODUCTION

There has been considerable interest in the mapping of linear canonical transformations in coordinate-momentum phase space to unitary operators[1] using the "Integration Within Ordered Product" (IWOP) technique[2]. The attractive point for these unitary operators is that they are represented by normally ordered forms of the squeeze operator[3, 4]. Since squeeze operators form the SU (1, 1) Lie algebra, there is both a conceptual and computational advantage for this form of unitary operator. As a consequence, the Feynman propagator for linear transformations in coordinate-momentum phase space is easily derived[5] using this unitary operator. In addition, we realized in the course of the calculation that the Feynman propagator is written in the form of the expo-nential of the classical generating function which engenders the linear canonical transformations in coordinate-momentum phase space. This indicates a new correspondence between classical and quantum mechanics.

In classical mechanics, it is well known[6] that there are four types of generating function which generate the same canonical transformations. The difference between the four is the choice of canonical variable used to form the generating function. The question arises as to what prop-agators are appropriate for quantum mechanics that corresponding to the situation in classical mechanics? The purpose of this paper is to show that the four types of transformation func-tion which correspond to the four types of generating function are derived from the same unitary operator, which is the normally ordered form of the squeeze operator.

In the next section, we first review the canonical transformations which cause the linear canonical transformations in coordinate-momentum phase space. We show four types of generating function. In section Ⅲ, we derive the four types of transformation function corresponding to the four types of generating function from the same unitary operator which is the normally ordered form of the squeeze operator. Section Ⅳ is devoted to a discussion.

Ⅱ. LINEAR CANONICAL TRANSFORMATIONS

Linear canonical transformations in coordinate-momentum phase space with time-dependent coefficients are defined by

$$Q(t) = A(t)q + B(t)p, \tag{1a}$$

$$P(t) = C(t)q + D(t)p, \tag{1b}$$

where (q, p) are the old canonical variables which describes the position and momentum at intial time and (Q, P) are the new canonical variables which describes the position and momentum at later time. The coefficients $A(t)$, $B(t)$, $C(t)$ and $D(t)$ designate the linear canonical transformations and the real functions of time t. In order for these transformations to be canonical, the coefficients $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are constrained by the Poisson bracket,

$$\begin{aligned} [Q(t), P(t)]_c &= \frac{\partial Q(t)}{\partial q}\frac{\partial P(t)}{\partial p} - \frac{\partial P(t)}{\partial q}\frac{\partial Q(t)}{\partial p}, \\ &= A(t)D(t) - B(t)C(t) = 1. \end{aligned} \tag{2}$$

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Table I. Generating functions which engender linear canonical transformations (1).

	Q	P
q	$W_1=\frac{qQ}{B(t)}-\frac{A(t)}{2B(t)}q^2-\frac{D(t)}{2B(t)}Q^2$	$W_2=\frac{qP}{D(t)}-\frac{C(t)}{2D(t)}q^2+\frac{B(t)}{2D(t)}P^2$
p	$W_3=-\frac{pQ}{A(t)}+\frac{B(t)}{2A(t)}p^2-\frac{C(t)}{2A(t)}Q^2$	$W_4=-\frac{pP}{C(t)}+\frac{D(t)}{2C(t)}p^2+\frac{A(t)}{2C(t)}P^2$

These linear canonical transformations (1) in coordinate-momentum phase space are derived from the type-1 generating function $W_1(q,Q,t)$ as

$$W_1(q,Q,t)=\frac{qQ}{B(t)}-\frac{A(t)}{2B(t)}q^2-\frac{D(t)}{2B(t)}Q^2, \tag{3}$$

with the ordinary prescription:

$$p=\frac{\partial W_1}{\partial q}, \quad P=-\frac{\partial W_1}{\partial Q}. \tag{4}$$

It is well known in classical mechanics[6] that since the generating function can be written as a function of one old and one new canonical variable, there are four types of generating function which generate the same canonical transformation (1). A list of other types of generating function is provided in Table 1. These four generating functions are related to each other by a Legendre transformation.

Corresponding to generating functions of type-2, 3 and 4, the derivatives of them are established:

$$W_2(q,P,t) \quad \text{for} \quad p=\frac{\partial W_2}{\partial q}, \quad Q=\frac{\partial W_2}{\partial P}, \tag{5}$$

$$W_3(p,Q,t) \quad \text{for} \quad q=-\frac{\partial W_3}{\partial p}, \quad P=-\frac{\partial W_3}{\partial Q}, \tag{6}$$

$$W_4(p,P,t) \quad \text{for} \quad q=-\frac{\partial W_4}{\partial p}, \quad Q=\frac{\partial W_4}{\partial P}, \tag{7}$$

which result in linear canonical transformations (1).

Ⅲ. SQUEEZE OPERATOR AND TRANSFORMATION FUNCTIONS

In this section, we shall derive the transformation functions. From now on, to distinguish the c-number variables, we attach ^ for the q-number variables. The unitary operator $\hat{U}(t)$ which causes a linear transformation in coordinate-momentum phase space is written as the normally ordered form of the squeeze operator[1, 5]

$$\hat{U}(t)=\exp\left[-\frac{r(t)}{s(t)}\hat{K}_+\right]\exp\left[-2\hat{K}_0\ln s(t)\right] \times\exp\left[\frac{r^*(t)}{s(t)}\hat{K}_-\right], \tag{8}$$

where $\hat{K}\pm$ and $\hat{K}_0$ are defined in terms of the annihilation $\hat{a}=(\hat{q}+i\hat{p})\sqrt{2}$ and creation $\hat{a}(\hat{q}-i\hat{p})\sqrt{2}$ operators as,

$$\hat{K}_+=\frac{\hat{a}^\dagger\hat{a}^\dagger}{2}, \quad \hat{K}_0=\frac{\hat{a}^\dagger\hat{a}}{2}+\frac{1}{4}, \quad \hat{K}_-=\frac{\hat{a}\hat{a}}{2}. \tag{9}$$

These operators form the SU (1, 1) Lie algebra,

$$\left[\hat{K}_+,\hat{K}_-\right]=-2\hat{K}_0, \quad \left[\hat{K}_0,\hat{K}_\pm\right]=\pm\hat{K}_\pm, \tag{10}$$

where $[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}$ is a commutator between two operators. Transforming the annihilation and creation operators with (8) such as $\hat{U}^\dagger(t)\hat{a}\hat{U}(t)$ and $\hat{U}^\dagger(t)\hat{a}^\dagger\hat{U}(t)$ we obtain the linear transformation in coordinate-momentum phase space

$$\hat{Q}(t)=\hat{U}^\dagger(t)\hat{q}\hat{U}(t)=A(t)\hat{q}+B(t)\hat{p}, \tag{11a}$$

$$\hat{P}(t)=\hat{U}^\dagger(t)\hat{p}\hat{U}(t)=C(t)\hat{q}+D(t)\hat{p}, \tag{11b}$$

with

$$\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t)\end{pmatrix}=\begin{pmatrix}\frac{s+s^*-r-r^*}{2} & \frac{-is+is^*+ir-ir^*}{2} \\ \frac{is-is^*+ir-ir^*}{2} & \frac{s+s^*+r+r^*}{2}\end{pmatrix}. \tag{12}$$

Now, we calculate the four types of transformation function: $<Q|\hat{U}(t)|q>$, $<P|\hat{U}(t)|q>$, $<Q|\hat{U}(t)|p>$, $<P|\hat{U}(t)|p>$. The coherent state $|z>$ is defined by the eigenstate of the annihilation operator $\hat{a}$ with the complex eigenvalue z, i.e.,

$$\hat{a}|z>=z|z>, \tag{13}$$

and forms the completeness relation

$$\int\frac{d^2z}{2\pi i}|z><z|=\int\frac{d[\mathrm{Re}(z)]d[\mathrm{Im}(z)]}{2\pi i}|z><z|=1. \tag{14}$$

We insert the identity operator (14) into the transformation function

$$<Q|\hat{U}(t)|q>=\int\frac{d^2z_1d^2z_2}{(2\pi i)^2}<Q|z_1><z_1|\hat{U}(t)|z_2><z_2|q>. \tag{15}$$

With the aid of the coherent-state representation of the unitary operator $\hat{U}(t)$ with arguments z_1 and z_2

$$<z_1|\hat{U}(t)|z_2>=\frac{1}{\sqrt{s}}\exp\left[-\frac{r}{2s}(z_1^*)^2+\frac{z_2z_1^*}{s}+\frac{r^*}{2s}(z_2)^2-\frac{|z_1|^2}{2}-\frac{|z_2|^2}{2}\right], \tag{16}$$

and the coordinate and momentum representation of the coherent state

Table 2. Transformation functions of the squeeze operator (8).

	Q	P
q	$< Q\|\hat{U}\|q >= \sqrt{\frac{1}{2\pi iB(t)}}\exp\left[-iW_1\right],$	$< P\|\hat{U}\|q >= \sqrt{\frac{1}{2\pi D(t)}}\exp\left[-iW_2\right]$
p	$< Q\|\hat{U}\|p >= \sqrt{\frac{1}{2\pi A(t)}}\exp\left[-iW_3\right]$	$< P\|\hat{U}\|p >= \sqrt{\frac{-1}{2\pi iC(t)}}\exp\left[-iW_4\right]$

$$< q|z > = \frac{1}{\pi^{1/4}}\exp\left[-\frac{q^2}{2}+\sqrt{2}zq-\frac{z^2}{2}-\frac{|z|^2}{2}\right], \qquad (17)$$

$$< p|z > = \frac{1}{\pi^{1/4}}\exp\left[-\frac{p^2}{2}-i\sqrt{2}zp+\frac{z^2}{2}-\frac{|z|^2}{2}\right], \qquad (18)$$

we integrate to obtain[5)]

$$< Q|\hat{U}(t)|q >= \sqrt{\frac{1}{2\pi iB(t)}}\exp\left[-iW_1(q,Q,t)\right], \qquad (19)$$

where $W_1(q, Q, t)$ is (3). We show an alternative derivation for $< Q|\hat{U}(t)|q >$, in the Appendix. The other transformation functions are derived in the same manner.

$$< P|\hat{U}(t)|q >= \sqrt{\frac{1}{\pi(s+s^*+r+r^*)}}\exp\left[\frac{-2iqP}{s+s^*+r+r^*} + \frac{q^2}{2}\frac{-s+s^*-r+r^*}{s+s^*+r+r^*}+\frac{P^2}{2}\frac{-s+s^*+r-r^*}{s+s^*+r+r^*}\right], \qquad (20)$$

$$= \sqrt{\frac{1}{2\pi D(t)}}\exp\left[-iW_2(q,P,t)\right], \qquad (21)$$

$$< Q|\hat{U}(t)|p >= \sqrt{\frac{1}{\pi(s+s^*-r-r^*)}}\exp\left[\frac{2ipQ}{s+s^*-r-r^*} + \frac{p^2}{2}\frac{-s+s^*+r-r^*}{s+s^*-r-r^*}+\frac{Q^2}{2}\frac{-s+s^*-r+r^*}{s+s^*-r-r^*}\right], \qquad (22)$$

$$= \sqrt{\frac{1}{2\pi A(t)}}\exp\left[-iW_3(p,Q,t)\right], \qquad (23)$$

$$< P|\hat{U}(t)|p >= \sqrt{\frac{-1}{\pi(-s+s^*-r+r^*)}}\exp\left[\frac{-2pP}{-s+s^*-r+r^*} + \frac{p^2}{2}\frac{s+s^*+r+r^*}{-s+s^*-r+r^*}+\frac{P^2}{2}\frac{s+s^*-r-r^*}{-s+s^*-r+r^*}\right], \qquad (24)$$

$$= \sqrt{\frac{-1}{2\pi iC(t)}}\exp\left[-iW_4(p,P,t)\right], \qquad (25)$$

where W_2, W_3 and W_4 are the generating functions which appear in Table 1. Corresponding to Table 1, we list all four transformation functions in Table 2. It is clear that these four functions are the exponentials of the generating functions which appear in Table 1. This result demonstrates a new correspondence between classical and quantum mechanics.

Whereas the generating functions in Table 1 are related by Legendre transformations, the transformation functions in Table 2 are related by Fourier transformations. For example, we shall derive the type-2 transformation function via type-1 transformation functions by

$$< P|\hat{U}|q >= \int dQ < P|Q >< Q|\hat{U}|q >, \qquad (26)$$

with

$$< P|Q >= \frac{1}{\sqrt{2\pi}}e^{-iPQ}, \qquad (27)$$

and the remaining task is a gaussian integration in terms of Q.

Ⅳ. DISCUSSION

We have obtained the transformation functions corresponding to linear canonical transformations in coordinate-momentum phase space. It was found that the transformation functions are expo-nentials of the generating functions. Originally, Dirac first discussed using the exponential of the classical generating function as a transformation function[7]. In that publication, Dirac used the type-2 transformation function (21). Later, he changed the canonical variable and discussed the type-1 transformation function (19)[8, 9]. These papers were cited by Feynman[10] in deriving his path integral form of quantum mechanics. However, these four types of transformation func-tion stand on equal footing as they are all derived from the same unitary operator (8) and are transformed by Fourier transformations.

Appendix

We define the eigenstate $|Q; t >$ of the operator $\hat{Q}(t)$ with eigenvalue Q to form the Q-representation:

$$\hat{Q}(t)|Q;t >= Q|Q;t > . \qquad (A.1)$$

By using (11a), the q-representation for the eigenstate $|Q; t >$ can be obtained from the differential equation

$$< q|\hat{Q}(t)|Q;t >= \left\{A(t)q - iB(t)\frac{\partial}{\partial q}\right\} < q|Q;t > = Q < q|Q;t > . \qquad (A.2)$$

Integrating (A.2) with the normalization condition

$$< Q;t|Q';t >= \delta(Q-Q'), \qquad (A.3)$$

we obtain

$$< q|Q;t > = \Phi(Q,t) \times \sqrt{\frac{1}{2\pi(-i)B(t)}} \times \exp\left[i\frac{2qQ - A(t)q^2}{2B(t)}\right], \qquad (A.4)$$

where $\Phi(Q, t)$ is an arbitrary phase factor which depends on Q and time t. Here, the meaning of $(-i)$ in the square root is clarified later. We consider the matrix element of the operator $\hat{P}(t)$ (11b) with an arbitrary state $|\Psi >$,

$$< Q;t|\hat{P}(t)|\Psi > = \int dq < Q;t|q >< q|\hat{P}(t)|\Psi >, \qquad (A.5)$$

$$= \int dq < Q;t|q > \left\{C(t)q - iD(t)\frac{\partial}{\partial q}\right\} < q|\Psi > . \qquad (A.6)$$

Integrating by parts and using the q-derivative of (A.4), we obtain

$$< Q;t|\hat{P}(t)|\Psi >= \left\{\frac{D(t)}{B(t)}Q - i\frac{\partial}{\partial Q} + \frac{i}{\Phi^*}\frac{\partial \Phi^*}{\partial Q}\right\} < Q;t|\Psi > . \qquad (A.7)$$

If we set

$$\Phi^*(Q,t) = \exp\left[i\frac{D(t)}{2B(t)}Q^2\right], \qquad (A.8)$$

$< Q; t\,|\hat{P}(t)|\,\Psi >$ is given by

$$< Q;t|\hat{P}(t)|\Psi >= -i\frac{\partial}{\partial Q} < Q;t|\Psi >, \qquad (A.9)$$

so that the q-representation of the eigenstate $|Q; t >$ turns out to be

$$< q|Q;t >= \sqrt{\frac{-1}{2\pi i B(t)}} \exp\left[i\left\{\frac{qQ}{B(t)} - \frac{A(t)}{2B(t)}q^2 - \frac{D(t)}{2B(t)}Q^2\right\}\right]. \qquad (A.10)$$

Since $|Q; t >$ is defined by a base ket, the time-dependency for this ket is defined as $|Q; t >= \hat{U}^\dagger(t)\,|Q >$. Thus, (A.10) becomes

$$< q|\hat{U}^\dagger(t)|Q >= \sqrt{\frac{-1}{2\pi i B(t)}} \exp\left[iW_1(q,Q,t)\right], \qquad (A.11)$$

which is the same result for (19)

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