論文　**Original Paper**

Moving Picture and Hamilton-Jacobi Theory in Quantum Mechanics

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Abstract: We propose a new picture, which we call the moving picture, in quantum mechanics. This picture provides the coordinate system which sticks to the particles. The Schrödinger equation in this picture is derived and its solution is examined. We also investigate the close relationship between the moving picture and the Hamilton-Jacobi theory in classical mechanics. This shows a new correspondence between classical and quantum mechanics. We examine the cases of the free particle and the harmonic oscillator, as an example of the usefulness of this picture.

Key words: Quantum mechanics, Canonical transformation, Hamilton-Jacobi equation

1. Introduction

Time development plays a fundamental role in quantum mechanics. In a general course on quantum mechanics, two well-known pictures are studied. One is the Schrödinger picture, and the other is the Heisenberg picture. In the former, the operators are fixed in time and the states vary with time, while in the latter it is vice versa. In both pictures, we use only a stationary base.

In contrast to this, we can also choose a set of base which acquires time de-pendency. The time-development of the base state is used in the path integral for-mulation of quantum mechanics. There, the Feynman propagator $K(x, t; x_0, t_0) = \langle x, t | x_0, t_0 \rangle$ is the corner stone and the moving frame is defined as $|x, t\rangle = e^{i\hat{H}t/\hbar}|x\rangle$. In this paper, we will take this picture, which we call moving picture, and reformu-late the quantum Hamilton formalism from this point of view.

Omote et al. [1] presented a new idea for the correspondence between classical and quantum mechanics. They proposed a new method for finding the solution to Schrödinger equation from a classical canonical transformation for the case in which the transformed Hamiltonian becomes zero. Under this transformation, they fixed the tranformation of the canonical position $q \to Q(t)$ and the momentum $p \to P(t)$. Next, they made the tranformed position operator $\hat{Q}(t)$ in the quantum mechanical sense, and the eigenstate $|Q, t\rangle$ of the operator $\hat{Q}(t)$ with eigenvalue Q form the set of base. In this representation, the Hamiltonian of the Schrödinger equation becomes zero in the same way as for the Hamilton-Jacobi theory in classical mechanics. Thus, they called their formulation the Hamilton-Jacobi picture and also found a solution to the Schrödinger equation. However, as we will show in this article, this Hamilton-Jacobi picture is nothing other than moving picture. In other words, their formulation is just a quantum mechanical formulation with respect to the moving frame.

Since the moving picture corresponds to looking at a moving body from a body-fixed moving reference frame, the transformed Hamiltonian always vanishes. This is a similar situation to the Hamilton-Jacobi theory in classical mechanics, where the canonical transformation makes the Hamiltonian become zero. The quantum Hamilton-Jacobi theory has been discussed by some authors[2, 3], but we will discuss more clearly the relationship between the moving picture in quantum mechanics and the Hamilton-Jacobi theory in classical mechanics.

This article is organized as follows. In section 2, we formulate the moving picture and fix our notation. The Schrödinger equation and its solution in the moving picture are derived in section 3, and some examples are discussed in section 4. In section 5, we will deduce the relationship between the moving picture and the Hamilton-Jacobi theory in classical mechanics. Section 6 is devoted to a discussion.

2. Moving Picture

Time development in quantum mechanics is governed by the Hamiltonian $\hat{H}(t)$ for the system with the Schrödinger equation for the time-evolution operator $\hat{T}(t)$,

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$$i\hbar\frac{\partial}{\partial t}\hat{T}(t,t_0)=\hat{H}(t)\hat{T}(t,t_0). \quad (1)$$

Here, t_0 is the initial time for the system and hereafter, for simplicity, $t_0=0$. The symbol ^ is attached on the operator in quantum mechanics.

When the solution to eq.(1) is found, we define the unitary transformation of the position $\hat{q}$ and the momentum $\hat{p}$ operators as

$$\begin{cases}\hat{Q}(t) &= \hat{T}(t)\hat{q}\hat{T}^{\dagger}(t),\\ \hat{P}(t) &= \hat{T}(t)\hat{p}\hat{T}^{\dagger}(t),\end{cases} \quad (2)$$

where the evolution of time is in the opposite direction to the Heisenberg operator. We note that since the right hand side is a function of position $\hat{q}$, momentum $\hat{p}$ and time t, the physical meaning of the left hand side, $\hat{Q}(t)$ and $\hat{P}(t)$, are not known at this point. It is easy to see that the transformed position and momentum operators have the commutation relation

$$\left[\hat{Q}(t),\hat{P}(t)\right]=i\hbar, \quad (3)$$

if $[\hat{q},\hat{p}]=i\hbar$ is satisfied. This is closely related to classical mechanics, where the Poisson bracket is kept to 1 before and after the canonical transformation.

Now we will make a complete set of base. We take the eigenstate $|Q;t\rangle$ of the operator $\hat{Q}(t)$ with eigenvalue Q to form the Q-representation:

$$\hat{Q}(t)|Q;t\rangle=Q|Q;t\rangle. \quad (4)$$

From the commutation relation eq.(3), the following relationship

$$\langle Q;t|\hat{P}(t)=-i\hbar\frac{\partial}{\partial Q}\langle Q;t|, \quad (5)$$

is satisfied. According to the transformation eq.(2), the time evolution of the base is defined by

$$|Q;t\rangle=\hat{T}(t)|Q\rangle, \quad \hat{q}|Q\rangle=Q|Q\rangle. \quad (6)$$

The q-representation of the eigenstate $|Q;t\rangle$ is calculated from eq.(4)

$$\langle q|\hat{Q}(t)|Q;t\rangle=Q\langle q|Q;t\rangle. \quad (7)$$

Since the transformed position operator $\hat{Q}(t)$ is a function of $\hat{q}$, $\hat{p}$ and t, this equation becomes the differential equation. With eq.(5) and the normalized condition

$$\langle Q;t|Q';t\rangle=\delta(Q-Q'), \quad (8)$$

we solve the differential equation (7) and get the q-representation of the eigenstate $|Q;t\rangle$

$$\langle q|Q;t\rangle. \quad (9)$$

This function will be used as the transformation function. We also note here that from eq.(6) this function can be written by

$$\langle q|Q;t\rangle=\langle q|\hat{T}(t)|Q\rangle, \quad (10)$$

which is the Feynman propagator from the "position" $|Q\rangle$ to the position $|q\rangle$. This situation will be accomplished by some examples in the later section.

3. Schrödinger equation and its solution

In the previous section, we have defined moving picture. Now we are ready to consider the Schrödinger equation in this picture.

3.1 Schrödinger equation

Let $|\Psi;t\rangle_{\mathrm{S}}$ be the state which is a solution of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi;t\rangle_{\mathrm{S}}=\hat{H}(t)|\Psi;t\rangle_{\mathrm{S}}, \quad (11)$$

where S stands for the Schrödinger picture. We define the wave function in the moving picture as

$$\Psi(Q,t)=\langle Q;t|\Psi;t\rangle_{\mathrm{S}}=\langle Q|\hat{T}^{\dagger}(t)|\Psi;t\rangle_{\mathrm{S}}. \quad (12)$$

Then, the Schrödinger equation becomes

$$i\hbar\frac{\partial}{\partial t}\Psi(Q,t)=\int dQ'\langle Q|\hat{K}(t)|Q'\rangle\Psi(Q',t), \quad (13)$$

where the transformed Hamiltonian $\hat{k}(t)$ in the moving picture is

$$\hat{K}(t)=\hat{T}^{\dagger}(t)\hat{H}(t)\hat{T}(t)+i\hbar\frac{\partial\hat{T}^{\dagger}(t)}{\partial t}\hat{T}(t). \quad (14)$$

But we notice that as with eq.(1), the transformed Hamiltonian $\hat{k}(t)$ is identically zero: $\hat{k}(t)=0$. This is a similar situation to the Hamilton-Jacobi theory in classical mechanics. That is why Omote et al. [1] have called this picture the Hamilton-Jacobi picture. However, this is nothing other than quantum mechanics with respect to the moving picture. It is certain that at first they seek a canonical transformation for which the Hamiltonian is zero in the region of classical mechanics, and then construct the representation for quantum mechanics. But it is unnecessary to go back to classical mechanics, as easily seen by the above argument.

3.2 Arbitrary state in moving picture

Since we have the transformation function eq.(9), the wave function for an arbitrary state $|\psi\rangle$ in the moving picture is easily obtained from

$$\langle Q;t|\psi\rangle=\int dq\langle Q;t|q\rangle\langle q|\psi\rangle, \quad (15)$$

if the q-representation of an arbitrary state $\langle q|\psi\rangle$ is known.

It is worth commenting here that the state in the moving picture is independent of time. In fact, if the arbitrary state in the Schrödinger picture is written by $|\psi;t\rangle_S$, its moving picture is

$$\langle Q;t|\psi;t\rangle_S = \langle Q|\hat{T}^\dagger(t)\hat{T}(t)|\psi\rangle_H = \langle Q|\psi\rangle_H, \tag{16}$$

where H stands for the Heisenberg picture. This means that the moving repre-sentation for an arbitrary state in the Schrödinger picture is equivalent to the Q-representation for an arbitrary state in the Heisenberg picture.

4. Examples

We are now in a position to apply the moving picture to some systems. We take two examples and restrict ourselves to cases where the Hamiltonian does not depend on time. In this case, the Schrödinger equation for the time-evolution operator eq.(1) is easily calculated and we get

$$\hat{T}(t) = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]. \tag{17}$$

However, the discussions in section 2 and 3 are applicable to all systems which satisfy the Schrödinger equation for the time-evolution operator eq.(1).

4.1 Free particle

The Hamiltonian of a free particle is

$$\hat{H} = \frac{\hat{p}^2}{2m}, \tag{18}$$

where m and p are the mass and momentum of the particle. In this case, the time-evolution operator is

$$\hat{T}(t) = \exp\left[-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t\right]. \tag{19}$$

It is easy to calculate the transformation of position and momentum,

$$\begin{cases} \hat{Q}(t) &= \hat{T}(t)\hat{q}\hat{T}^\dagger(t) = \hat{q} - \frac{t}{m}\hat{p}, \\ \hat{P}(t) &= \hat{T}(t)\hat{p}\hat{T}^\dagger(t) = \hat{p}. \end{cases} \tag{20}$$

From a classical mechanical point of view, this canonical transformation is repro-duced by the generating function

$$W(q,Q,t) = \frac{m}{2t}(q-Q)^2. \tag{21}$$

Furthermore, since this transformation is the Galilean transformation, the trans-formed Hamiltonian will vanish, as in the Hamilton-Jacobi theory. Also, since the operator $\hat{Q}$ is defined as a linear combination of $\hat{q}$ and $\hat{p}$, this transformation is not so simple transformation, like a point transformation.

Now we make up the set of base for the moving picture. From eq.(7) and eq.(20), we obtain the differential equation

$$\langle q|\hat{Q}(t)|Q;t\rangle = \left[q + i\hbar\frac{t}{m}\frac{\partial}{\partial q}\right]\langle q|Q;t\rangle = Q\langle q|Q;t\rangle. \tag{22}$$

Integrating this equation, we have

$$\langle q|Q;t\rangle = \Phi(Q)\sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{i}{\hbar}\frac{m}{t}\left(\frac{q^2}{2} - qQ\right)\right], \tag{23}$$

where $\Phi(Q)$ is an arbitrary phase factor. On the other hand, the matrix element of an arbitrary state $|\psi\rangle$ is

$$\langle Q;t|\hat{P}|\psi\rangle = \int dq\langle Q;t|q\rangle\langle q|\hat{P}|\psi\rangle, \tag{24}$$

$$= \left[-i\hbar\frac{\partial}{\partial Q} + \frac{i\hbar}{\Phi^*(Q)}\frac{\partial\Phi^*(Q)}{\partial Q} - \frac{m}{t}Q\right] \times\langle Q;t|\psi\rangle. \tag{25}$$

If we take $\Phi^*(Q) = \exp\left[-\frac{i}{\hbar}\frac{m}{t}\frac{Q^2}{2}\right]$, the matrix element is given by

$$\langle Q;t|\hat{P}|\psi\rangle = -i\hbar\frac{\partial}{\partial Q}\langle Q;t|\psi\rangle, \tag{26}$$

and the transformation function turns out to be

$$\langle q|Q;t\rangle = \langle q|\hat{T}(t)|Q\rangle = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{i}{\hbar}\frac{m}{2t}(q-Q)^2\right]. \tag{27}$$

It is of interest that this transfromation function is nothing but the Feynman prop-agator from the "position" $|Q\rangle$ to the position $|q\rangle$.

As an example of an arbitrary state, we take a momentum eigenstate $|p\rangle$ with eigenvalue p, which satisfies

$$\hat{p}|p\rangle = p|p\rangle. \tag{28}$$

We calculate

$$\langle Q;t|p\rangle = \int dq\langle Q;t|q\rangle\langle q|p\rangle = \langle Q|\hat{T}^\dagger(t)|p\rangle, \tag{29}$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\exp\left[\frac{i}{\hbar}Qp + \frac{i}{\hbar}\frac{p^2}{2m}t\right]. \tag{30}$$

4.2 Harmonic Oscillator

The Hamiltonian of the harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2, \tag{31}$$

where m and ω are the mass and frequency of the particle. Since this Hamiltonian is independent of time, the time-evolution operator is given by

$$\hat{T}(t) = \exp\left[-\frac{i}{\hbar}\left\{\frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2\right\}t\right]. \tag{32}$$

In the same manner, it is easy to calculate the transformed position and momen-tum,

$$\begin{cases} \hat{Q}(t) &= \hat{T}(t)\hat{q}\hat{T}^\dagger(t) \\ \hat{P}(t) &= \hat{T}(t)\hat{p}\hat{T}^\dagger(t) \end{cases} \tag{33}$$

From the classical mechanical point of view, this transformation is deduced from the generating function

$$W(q,Q,t)=\frac{m\omega}{\sin\omega t}\left(\frac{q^2+Q^2}{2}\cos\omega t-qQ\right). \tag{34}$$

Furthermore, since this transformation refers to a rotational system in phase space, the transformed Hamiltonian will vanish, as with the Hamilton-Jacobi theory. More-over, since the operator $\hat{Q}$ is defined as a linear combination of $\hat{q}$ and $\hat{p}$, this trans-formation is not so simple transformation, like a point transformation.

Now we construct the set of base for the moving picture. From eq.(7) and eq.(33), we obtain the differential equation

$$\langle q|\hat{Q}(t)|Q;t\rangle=\left[q\cos\omega t+\frac{i\hbar}{m\omega}\sin\omega t\frac{\partial}{\partial q}\right]\langle q|Q;t\rangle=Q\langle q|Q;t\rangle. \tag{35}$$

Integrating this equation, we have

$$\langle q|Q;t\rangle=\Phi(Q)\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}\exp\left[\frac{i}{\hbar}\frac{m\omega}{\sin\omega t}\left(\frac{q^2}{2}\cos\omega t-qQ\right)\right], \tag{36}$$

where $\Phi(Q)$ is an arbitrary phase factor. On the other hand, the matrix element of an arbitrary state ψ is

$$\langle Q;t|\hat{P}|\psi\rangle = \int dq\langle Q;t|q\rangle\langle q|\hat{P}|\psi\rangle, \tag{37}$$

$$=\left[-i\hbar\frac{\partial}{\partial Q}+\frac{i\hbar}{\Phi^*(Q)}\frac{\partial\Phi^*(Q)}{\partial Q}-m\omega Q\frac{\cos\omega t}{\sin\omega t}\right]\langle Q;t|\psi\rangle. \tag{38}$$

If we take $\Phi^*(Q)=\exp\left[-\frac{i}{\hbar}\frac{m\omega\cos\omega t}{2\sin\omega t}Q^2\right]$, the matrix element is given by

$$\langle Q;t|\hat{P}|\psi\rangle=-i\hbar\frac{\partial}{\partial Q}\langle Q;t|\psi\rangle, \tag{39}$$

and the transformation function turns out to be

$$\langle q|Q;t\rangle=\langle q|\hat{T}(t)|Q\rangle=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}\times\exp\left[\frac{i}{\hbar}\frac{m\omega}{\sin\omega t}\left(\frac{q^2+Q^2}{2}\cos\omega t-qQ\right)\right]. \tag{40}$$

It is of interest that this transformation function is nothing other than the Feynman propagator from the "position" $|Q\rangle$ to the position $|q\rangle$.

As an example of an arbitrary state, we take a number eigenstate $|n\rangle$, which satisfies

$$\hat{N}|n\rangle=n|n\rangle, \tag{41}$$

where the number operator is defined by $\hat{N}=\hat{a}^\dagger\hat{a}$. The operator $\hat{a}$ is related to the position and the momentum operators as $\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right)$.

In order to obtain the Q-representation $\langle Q;t|n\rangle$, it is easier to proceed as follows rather than via eq.(15). We first calculate

$$\langle Q;t|\hat{a}^\dagger|\psi\rangle=\int dq\langle Q;t|q\rangle\langle q|\hat{a}^\dagger|\psi\rangle, \tag{42}$$

$$=\sqrt{\frac{m\omega}{2\hbar}}\int dq\langle Q;t|q\rangle\left(q-\frac{\hbar}{m\omega}\frac{\partial}{\partial q}\right)\langle q|\psi\rangle, \tag{43}$$

which can be rewritten as

$$\langle Q;t|\hat{a}^\dagger|\psi\rangle=\sqrt{\frac{m\omega}{2\hbar}}e^{i\omega t}\left(-\frac{\hbar}{m\omega}\right)\exp\left[\frac{m\omega}{2\hbar}Q^2\right]\frac{\partial}{\partial Q}\times\left\{-\exp\left[\frac{m\omega}{2\hbar}Q^2\right]\langle Q;t|\psi\rangle\right\}. \tag{44}$$

Further, a straight cast is done to get

$$\langle Q;t|0\rangle = \int\langle Q;t|q\rangle\langle q|0\rangle, \tag{45}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}Q^2+i\frac{\omega}{2}t\right], \tag{46}$$

thus we have

$$\langle Q;t|n\rangle = \frac{1}{\sqrt{n!}}\langle Q;t|\left(\hat{a}^\dagger\right)^n|0\rangle, \tag{47}$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\exp\left[-\frac{m\omega}{2\hbar}Q^2+i\left(n+\frac{1}{2}\right)\omega t\right]\times \mathrm{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}Q\right), \tag{48}$$

where $\mathrm{H}_n(\xi)$ is the Hermite polynomial of argument ξ.

The other example is a coherent state $\hat{a}|z\rangle=z|z\rangle$. The Q-representation is

$$\langle Q;t|z\rangle = \sum_{n=0}^{\infty}\langle Q;t|n\rangle\langle n|z\rangle, \tag{49}$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}Q^2+2zQe^{i\omega t}\sqrt{\frac{m\omega}{2\hbar}}-\frac{z^2}{2}e^{2i\omega t}-\frac{|z|^2}{2}+i\frac{\omega t}{2}\right]. \tag{50}$$

5. Principal function and moving picture

Thus far, we have been discussing quantum mechanics from the perspective of the moving picture. In this case, since we are looking at the moving body from the body-fixed moving reference frame, the transformed Hamiltonian $\hat{k}(t)$ always vanishes. The situation is similar for the Hamilton-Jacobi theory in classical mechanics. In this section, we will study the moving picture from a different point of view. We start from the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(q,t)=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}+V(q)\right]\psi(q,t). \tag{51}$$

Let us write the wave function as

$$\psi(q,t)=\exp\left[\frac{i}{\hbar}S(q,t)\right], \tag{52}$$

and putting this into the above Schrödinger equation, we get

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) + \frac{\partial S}{\partial t} - \frac{i\hbar}{2m}\frac{\partial^2 S}{\partial q^2} = 0. \tag{53}$$

This form of the equation has been well studied in the classical limit using the WKB formalism where the focus is on the stationary-state solution. Here, we consider this equation from a different point of view. We define the new function[4)]

$$F \equiv \frac{1}{2m}\frac{\partial^2 S}{\partial q^2}. \tag{54}$$

If the function S is given by the polynomial with respect to q up to second order, F is independent of q and depends only on t. Hereafter, we consider this case and define the new function W as

$$S(q,t) = W(q,t) + i\hbar \int^t dt' F(t'). \tag{55}$$

Putting this back into eq.(53), we get

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + V(q) + \frac{\partial W}{\partial t} = 0. \tag{56}$$

This is the Hamilton-Jacobi equation that appears in classical mechanics and W is a Hamilton's principal function.

To sum up the above argument, once we have the solution W of eq.(56), we derive the function F from eq.(54) and also derive the function S from eq.(55). Accordingly, we get the solution to the Schrödinger equation eq.(51). Two examples will be discussed below.

5.1 Free particle

Eq.(21) is the solution of the Hamilton-Jacobi equation in classical mechanics,

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{\partial W}{\partial t} = 0. \tag{57}$$

From eqs.(21), (54) and (55), the function F is obtained by

$$F = \frac{1}{2m}\frac{\partial^2 S}{\partial q^2} = \frac{1}{2m}\frac{\partial^2 W}{\partial q^2} = \frac{1}{2t}. \tag{58}$$

Putting this F back to (55), we obtain

$$S = W + i\hbar \ln\sqrt{t}. \tag{59}$$

The solution to the Schrödinger equation becomes

$$\psi = \exp\left[\frac{i}{\hbar}S\right] = \frac{1}{\sqrt{t}}\exp\left[\frac{i}{\hbar}\frac{m}{2t}(q-Q)^2\right]. \tag{60}$$

This is the transformation function eq.(27) except for the arbitrary constant $\sqrt{\frac{m}{2\pi i\hbar}}$ which is calculated from normalization of the wave function.

Next, we apply the Legendre transformation to eq.(21), where the variables are transformed from $W(q, Q, t)$ to $W(q, P, t)$,

$$\begin{aligned} W(q,P,t) &\equiv W(q,Q,t) + QP & (61)\\ &= qP - \frac{P^2}{2m}t. & (62)\end{aligned}$$

This equation is also the solution to the Hamilton-Jacobi equation(57). From eq.(54), the function F vanishes. Then the transformed solution (52) becomes

$$\psi = \langle q|P;t\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left[\frac{i}{\hbar}\left(qP - \frac{P^2}{2m}t\right)\right]. \tag{63}$$

The canonical transformation in quantum mechanics was studied in the early days of quantum mechanics[5)], and has recently been reconsidered by some authors[2, 3)]. As already pointed out in[3)], it is interesting that the generating functions eq.(21) and eq.(62) are transformed by the Legendre transformation, while the wave functions eq.(27) and eq.(63) are transformed by the Fourier transformation, which is easily accomplished by

$$\begin{aligned} \langle q|P;t\rangle &= \int dQ \langle q|Q;t\rangle\langle Q;t|P;t\rangle, & (64)\\ &= \int dQ\langle q|Q;t\rangle \times \frac{1}{\sqrt{2\pi\hbar}}e^{iPQ/\hbar}. & (65)\end{aligned}$$

5.2 Harmonic Oscillator

Eq.(34) is the solution of the Hamilton-Jacobi equation in classical mechanics,

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{m\omega^2}{2}q^2 + \frac{\partial W}{\partial t} = 0. \tag{66}$$

From eqs.(21), (54) and (55), the function F is obtained by

$$F = \frac{1}{2m}\frac{\partial^2 S}{\partial q^2} = \frac{1}{2m}\frac{\partial^2 W}{\partial q^2} = \frac{\omega}{2}\cos\omega t. \tag{67}$$

Putting this F back to (55), we obtain

$$S = W + i\hbar\ln\sqrt{\sin\omega t}. \tag{68}$$

The solution to the Schrödinger equation is

$$\psi = \frac{1}{\sqrt{\sin\omega t}}\exp\left[\frac{i}{\hbar}\frac{m\omega}{\sin\omega t}\left(\frac{q^2+Q^2}{2}\cos\omega t - qQ\right)\right]. \tag{69}$$

This is the transformation function eq.(40) except for the arbitrary constant $\sqrt{\frac{m\omega}{2\pi i\hbar}}$ which is calculated from normalization of the wave function.

In a similar manner, we apply the Legendre transformation: $W(q, Q, t) \rightarrow W(q, P, t)$, and obtain

$$\begin{aligned} W(q,P,t) &\equiv W(q,Q,t) + QP, & (70)\\ &= \frac{qP}{\cos\omega t} - \left(\frac{m\omega^2}{2}q^2 + \frac{P^2}{2m}\right)\frac{\tan\omega t}{\omega}. & (71)\end{aligned}$$

From this equation, the function F becomes

$$F = \frac{d}{dt} \ln \sqrt{\cos \omega t} \tag{72}$$

then the solution to the transformed Schrödinger equation becomes

$$\psi = \langle q|P;t\rangle = \frac{1}{\sqrt{2\pi\hbar\cos\omega t}} \times \exp\left[\frac{i}{\hbar}\left\{\frac{qP}{\cos\omega t} - \left(\frac{m\omega^2}{2}q^2 + \frac{P^2}{2m}\right)\frac{\tan\omega t}{\omega}\right\}\right]. \tag{73}$$

The same argument for the transformation is also applied to the Harmonic Os-cillator case. That is, the generating functions eq.(34) and eq.(71) are transformed by the Legendre transformation, while the wave functions eq.(40) and eq.(73) are transformed by the Fourier transformation.

6. Discussion

We have investigated the moving picture in quantum mechanics and could clearly formulate a representation of this picture.

In contrast to the Heisenberg and Schrödinger pictures, we found that the trans-formed Hamiltonian becomes zero. This is similar to the case of the Hamilton-Jacobi theory in classical mechanics. It is unsuitable to use the Hamilton-Jacobi represen-tation terminology for this picture because the Hamilton-Jacobi theory permits a wider variety of transformations than in the quantum case. For example, we take a harmonic oscillator. The following transformation

$$\begin{cases} Q &= \frac{1}{\omega}\tan^{-1}\left(\frac{m\omega q}{p}\right) - t, \\ P &= \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2, \end{cases} \tag{74}$$

is really a canonical transformation and its transformed Hamiltonian becomes zero. But in this case, we never formulate a quantum representation for this transforma-tion, because the operators $\hat{q}$ and $\hat{p}$ are included in an arctangent function.

We also discussed the relationship between the moving picture and the Hamilton-Jacobi theory in classical mechanics. This shows a new correspondence between classical and quantum mechanics.

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