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On the torsion units of ZD_n

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Abstract: For a finite group G, we denote by ZG (resp. QG) the integral group ring (resp. rational group algebra) of G, and by $U(ZG)$ (resp. $U(QG)$) the unit group of ZG (resp. QG). In this paper, we will give torsion units in $U(ZG)$ which are not conjugate to trivial units in $U(ZG)$ but are conjugate to them in $U(QG)$.

Key words: integral group ring, unit group of ZG, torsion units

1. Introduction

For a ring S, we denote by $U(S)$ the unit group of S. We say that the element u in $U(S)$ is the torsion unit if the order of u is finite.

For a finite group G, we denote by ZG the integral group ring of G. Then, obviously, $\pm g\,(\in ZG)$ are torsion units in $U(ZG)$. These units are called trivial units in $U(ZG)$.

In this area, the following theorem is well-known：

Theorem 1. (c.f. [3])

Let G be a finite group. Then ZG has only trivial units if and only if G is an abelian group of exponent 2, 3, 4, 6 or $G \cong E \times Q_8$, where Q_8 is the quaternion group of order 8 and E is an elementary abelian 2-group.

We denote by C_n the cyclic group of order n, and D_n the dihedral group of order $2n$.

Throughout this paper, Q, Z and N denote the set of rational numbers, the set of rational integers and the set of natural numbers, respectively.

The purpose of this paper is to give torsion units in ZD_n which are conjugate to the trivial units in $U(QD_n)$, but are not conjugate to them in $U(ZD_n)$.

2. Cyclic algebras and representation

Let L be a finite Galois extension of the field K, with the Galois group $G(L/K)$. Suppose that $G(L/K)$ is the cyclic group of order n, with the generator τ,

$$G(L/K) \cong C_n = \langle \tau \rangle,\ \tau^n = 1.$$

Then we can construct the following cyclic algebra

$$(L/K, u, a) = Lu^0 \oplus Lu^1 \oplus \cdots \oplus Lu^{n-1},$$

where $(L/K, u, a)$ is the L-vector space of dimension n, with the basis $\{u^0, u^1, \cdots, u^{n-1}\}$, and the relations $u^n = a$ and $ulu^{-1} = \tau(l)$, for $l \in L$. Here, we identify u^0 with the unity element of $(L/K, u, a)$, so we can identify Lu^0 with L. Hence we may assume that $L \subset (L/K, u, a)$.

Then the following theorem is well-known：

Theorem 2. (c.f. [2])

(1) $(L/K, u, a)$ is a central simple K-algebra.

(2) $(L/K, u, a) \cong (L/K, u^s, a^s)$ for each $s \in$ Z such that $(s, n) = 1$.

(3) $(L/K, u, a) \cong M_n(K)$, if and only if $a \in N_{L/K}(L^*)$, where $N_{L/K}$ is the norm map and $L^* = U(L)$. In particular, when $a = 1$, $(L/K, u, 1) \cong M_n(K)$.

In the case when $a = 1$, $(L/K, u, 1)\bar{u} = L\bar{u}$ is the irreducible $(L/K, u, 1)$- module, where $\bar{u} = u^0 + u^1 + \cdots + u^{n-1}$. The left action of $(L/K, u, 1)$ is the following

$$(l_1u^0 + l_2u^1 + \cdot + l_{n-1}u^{n-1})x\bar{u} = (l_1x + l_2\tau(x) + \cdots + l_{n-1}\tau^{n-1}(x))\bar{u},$$

where, $l_1u^0 + l_2u^1 + \cdots + l_{n-1}u^{n-1} \in (L/K, u, 1)$ and $x\bar{u} \in L\,\bar{u}$.

Using this module, we can construct the matrix representation of $(L/K, u, 1)$,

$$T : (L/K, u, 1) \to M_n(K).$$

For a positive integer n, we denote by ζ_n the primitive n-th root of unity. and by Q(ζ_n) the n-th cyclotomic field.

Let $Q(\zeta_n + \zeta_n^{-1})$ be the maximal real subfield of Q(ζ_n). Then Q(ζ_n)/Q$(\zeta_n + \zeta_n^{-1})$ is the cyclic extension and

$$\mathrm{Gal}(Q(\zeta_n)/Q(\zeta_n + \zeta_n^{-1})) \cong C_2 = \langle \tau \rangle,\ \tau^2 = 1.$$

Using this field extension, we can construct the cyclic algebra

$$(Q(\zeta_n)/Q(\zeta_n + \zeta_n^{-1}), \tau, 1) = Q(\zeta_n) \oplus Q(\zeta_n)u$$

where $u^2 = 1$, and $u\zeta_n u^{-1} = \tau(\zeta_n) = \zeta_n^{-1}$,

Then $(Q(\zeta_n)/Q(\zeta_n + \zeta_n^{-1}), \tau, 1)\bar{u} = Q(\zeta_n)\bar{u} = Q(\zeta_n)(1 + u)$ is the irreducible $(Q(\zeta_n)/Q(\zeta_n + \zeta_n^{-1}), \tau, 1)$-module with

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the left action $\zeta_n(x\bar{u}) = \zeta_n x\bar{u}$ and $u(x\bar{u}) = \tau(x)\bar{u}$,
where $\zeta_n, u \in (Q(\zeta_n)/Q(\zeta_n+\zeta_n^{-1}), \tau, 1)$ and $x\bar{u} \in Q(\zeta_n)\bar{u}$.
Note that $Q(\zeta_n)$ can be considered as $Q(\zeta_n+\zeta_n^{-1})$- vector space of dimension 2. We can choose its basis $\{1, \zeta_n\}$. So, we have

$$Q(\zeta_n) = Q(\zeta_n+\zeta_n^{-1}) \oplus Q(\zeta_n+\zeta_n^{-1})\zeta_n.$$

Using this basis, we can construct the matrix representation

$$T: (Q(\zeta_n)/Q(\zeta_n+\zeta_n^{-1}), \tau, 1) \to M_2(Q(\zeta_n+\zeta_n^{-1})),$$

where

$$T(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T(\zeta_n) = \begin{pmatrix} 0 & -1 \\ 1 & \zeta_n + \zeta_n^{-1} \end{pmatrix},$$

$$T(\tau) = \begin{pmatrix} 1 & \zeta_n + \zeta_n^{-1} \\ 0 & -1 \end{pmatrix}, \qquad T(\zeta_n\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

And

$$T(a_0 + a_1\zeta_n + b_0\tau + b_1\zeta_n\tau) = \begin{pmatrix} a_0 + b_0 & -a_1 + b_0(\zeta_n + \zeta_n^{-1}) + b_1 \\ a_1 + b_1 & a_0 + a_1(\zeta_n + \zeta_n^{-1}) - b_0 \end{pmatrix},$$

where, $a_0, a_1, b_0, b_1, \in Q(\zeta_n+\zeta_n^{-1})$.

3. Structure of $\mathbf{Q}D_n$

Let n and d be the positive integers such that d is the divizor of n and $3 \leqq d$.

Write $C_n = \langle\sigma\rangle$, and $D_n = \langle\sigma, \tau \mid \sigma^n = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\rangle$.
Then there is an Q-algebra isomorphism

$$\mathbf{Q}C_n/(\Phi_d(\sigma)\mathbf{Q}C_n) \cong \mathbf{Q}(\zeta_d)$$

where σ is a generator of C_n, i.e. $C_n = \langle\sigma\rangle$, and $\Phi_d(X)$ is the d-th cyclotomic polynomial. And

$$\mathbf{Q}C_n \cong \bigoplus_{d|n} \mathbf{Q}C_n/(\Phi_d(\sigma)\mathbf{Q}C_n) \cong \bigoplus_{d|n} \mathbf{Q}(\zeta_d).$$

Using these isomorphisms, we have

$$\mathbf{Q}D_n/(\Phi_d(\sigma)\mathbf{Q}D_n) \cong (\mathbf{Q}(\zeta_d)/\mathbf{Q}(\zeta_d + \zeta_d{}^{-1}), \tau, 1),$$

for $3 \leqq$ d, and

$$\mathbf{Q}D_n \cong \bigoplus_{d|n} \mathbf{Q}D_n/(\Phi_d(\sigma)\mathbf{Q}D_n).$$

Therefore, when n is odd,

$$\begin{aligned}\mathbf{Q}D_n &\cong \bigoplus_{d|n,\ 3\leqq d} (\mathbf{Q}(\zeta_d)/\mathbf{Q}(\zeta_d + \zeta_d^{-1}), \tau, 1) \bigoplus \mathbf{Q}\langle\tau\rangle \\ &\cong \bigoplus_{d|n,\ 3\leqq d} (\mathbf{Q}(\zeta_d)/\mathbf{Q}(\zeta_d + \zeta_d^{-1}), \tau, 1) \oplus \mathbf{Q} \oplus \mathbf{Q} \\ &\cong \bigoplus_{d|n,\ 3\leqq d} M_2(\mathbf{Q}(\zeta_d + \zeta_d^{-1})) \oplus \mathbf{Q} \oplus \mathbf{Q}.\end{aligned}$$

And, when n is even

$$\begin{aligned}\mathbf{Q}D_n &\cong \bigoplus_{d|n,\ 3\leqq d} (\mathbf{Q}(\zeta_d)/\mathbf{Q}(\zeta_d + \zeta_d^{-1}), \tau, 1) \bigoplus \mathbf{Q}(\langle\sigma/\sigma^2\rangle \times \langle\tau\rangle) \\ &\cong \bigoplus_{d|n,\ 3\leqq d} (\mathbf{Q}(\zeta_d)/\mathbf{Q}(\zeta_d + \zeta_d^{-1}), \tau, 1) \oplus \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} \\ &\cong \bigoplus_{d|n,\ 3\leqq d} M_2(\mathbf{Q}(\zeta_d + \zeta_d^{-1})) \oplus \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q}\end{aligned}$$

where, we use the isomorphism

$$(\mathbf{Q}(\zeta_d)/\mathbf{Q}(\zeta_d + \zeta_d^{-1}), \tau, 1) \cong M_2(\mathbf{Q}(\zeta_d + \zeta_d^{-1}))$$

as $\mathbf{Q}(\zeta_d + \zeta_d^{-1})$-algebras.

Therefore, when n is odd, there is a natural monomorphism

$$\begin{aligned}ZD_n &\longrightarrow \bigoplus_{d|n,\ 3\leqq d} (Z[\zeta_d]/Z[\zeta_d + \zeta_d^{-1}], \tau, 1) \bigoplus Z\langle\tau\rangle \\ &\subseteq \bigoplus_{d|n,\ 3\leqq d} (Z[\zeta_d]/Z[\zeta_d + \zeta_d^{-1}], \tau, 1) \oplus Z \oplus Z \\ &\subseteq \bigoplus_{d|n,\ 3\leqq d} M_2(Z[\zeta_d + \zeta_d^{-1}]) \oplus Z \oplus Z.\end{aligned}$$

When n is even, there is a natural monomorphism

$$\begin{aligned}ZD_n &\longrightarrow \bigoplus_{d|n,\ 3\leqq d} (Z[\zeta_d]/Z[\zeta_d + \zeta_d^{-1}], \tau, 1) \bigoplus Z(\langle\sigma/\sigma^2\rangle \times \langle\tau\rangle) \\ &\subseteq \bigoplus_{d|n,\ 3\leqq d} (Z[\zeta_d]/Z[\zeta_d + \zeta_d^{-1}], \tau, 1) \oplus Z \oplus Z \oplus Z \oplus Z \\ &\subseteq \bigoplus_{d|n,\ 3\leqq d} M_2(Z[\zeta_d + \zeta_d^{-1}]) \oplus Z \oplus Z \oplus Z \oplus Z.\end{aligned}$$

4. Construction of units in ZG

In [5], we have given a method of construction of nontrivial units. The result is the following；

Theorem 3 （[5]）.
Let G be a finite group. Take a unit u in $\mathbf{Z}G$ of order n and $f \in \mathbf{Z}G$ such that $f + f^{(1)} + f^{(2)} + \cdots + f^{(n-1)} = 0$, where $f^{(i)} = u^i f u^{-i}$.
Let c be a unit in $\mathbf{Z}G$ such that c is commutative with f and u, and set
$v = v(u, f, c) = f + (f^{(1)} + c)u + f^{(2)}u^2 + \cdots + f^{(n-1)}u^{n-1}$.
Then $v^n = c^n$. In particular, v is a unit in $\mathbf{Z}G$.

Proof. For the completeness of the paper, we will give a sketch of the proof. For $f_i, g_j \in ZG$, put $x = f_0 + f_1u + f_2u^2 + \cdots + f_{n-1}u^{n-1}$ and $y = g_0 + g_1u + g_2u^2 + \cdots + g_{n-1}u^{n-1}$. Then

$$xy = (f_0 + f_1u + f_2u^2 + \cdots + f_{n-1}u^{n-1})(g_0 + g_1u + g_2u^2 + \cdots + g_{n-1}u^{n-1}) = (f_0,\ f_1,\ f_2,\ \cdots, f_{n-1})W \begin{pmatrix} 1 \\ u \\ . \\ . \\ u^{n-1} \end{pmatrix},$$

where

$$W = \begin{pmatrix} g_0 & g_1 & g_2 \cdots & g_{n-1} \\ g_{n-1}^{(1)} & g_0^{(1)} & g_1 \cdots & g_{n-2}^{(1)} \\ \vdots & & & \\ g_1^{(n-1)} & g_2^{(n-1)} & g_3^{(n-1)} \cdots & g_0^{(n-1)} \end{pmatrix}.$$

Hence, for $v = v(u, f, c) = f + (f^{(1)} + c)u + f^{(2)}u^2 + \cdots$
$+ f^{(n-1)}u^{n-1}$,

we have

$$v^n = (f,\ f^{(1)} + c,\ f^{(2)},\ \cdots, f^{(n-1)})Y^{n-1} \begin{pmatrix} 1 \\ u \\ . \\ . \\ u^{n-1} \end{pmatrix},$$

where

$$Y = \begin{pmatrix} f & f^{(1)} + c & f^{(2)} \cdots & f^{(n-1)} \\ f & f^{(1)} & f^{(2)} + c \cdots & f^{(n-1)} \\ \vdots & & & \\ f & f^{(1)} & f^{(2)} \cdots & f^{(n-1)} + c \\ f + c & f^{(1)} & f^{(2)} \cdots & f^{(n-1)} \end{pmatrix}.$$

We can show

$$\begin{aligned} &(f,\ f^{(1)} + c,\ f^{(2)},\ \cdots, f^{(n-1)})Y^k \\ &= c^k(f,\ f^{(1)},\ \cdots f^{(k+1)} + c, \cdots f^{(n-1)}) \\ &+ c^k(f^{(n-1)},\ f,\ f^{(1)},\ \cdots, \cdots f^{(n-2)}) \\ &\vdots \\ &+ c^k(f^{(n-k)},\ f^{(n-k+1)},\ \cdots, \cdots f^{(n-k-1)}) \end{aligned}$$

for every k, $0 \leqq k \leqq n-1$, by the induction on k .
Here we identify $f^{(n)}$ with f.
When $k = n-1$, this equation implies that

$$(f,\ f^{(1)} + c,\ f^{(2)},\ \cdots, f^{(n-1)})Y^{n-1} = (c^n,\ 0,\ 0,\ \cdots, 0)$$

Hence, we have

$$v^n = (f,\ f^{(1)} + c,\ f^{(2)},\ \cdots, f^{(n-1)})Y^{n-1} \begin{pmatrix} 1 \\ u \\ . \\ . \\ u^{n-1} \end{pmatrix} = c^n .$$

Example 1

Let $D_n = \langle \sigma,\ \tau \mid \sigma^n = \tau^2 = 1,\ \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the dihedral group of order $2n$.

Set

$$f = a_1(\sigma - \sigma^{-1}) + a_2(\sigma^2 - \sigma^{-2}) + \cdots a_{\frac{n-1}{2}}(\sigma^{\frac{n-1}{2}} - \sigma^{-\frac{n-1}{2}})$$

and

$$\begin{aligned} v(\tau, f, 1) &= a_1(\sigma - \sigma^{-1}) + a_2(\sigma^2 - \sigma^{-2}) + \cdots a_{\frac{n-1}{2}}(\sigma^{\frac{n-1}{2}} \\ &- \sigma^{-\frac{n-1}{2}}) + (1 - a_1(\sigma - \sigma^{-1}) - a_2(\sigma^2 - \sigma^{-2}) - \cdots a_{\frac{n-1}{2}}(\sigma^{\frac{n-1}{2}} \\ &- \sigma^{-\frac{n-1}{2}}))\tau. \end{aligned}$$

when n is odd.
And set

$$f = a_1(\sigma - \sigma^{-1}) + a_2(\sigma^2 - \sigma^{-2}) + \cdots a_{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}} - \sigma^{-\frac{n-2}{2}})$$

and

$$\begin{aligned} v(\tau, f, 1) &= a_1(\sigma - \sigma^{-1}) + a_2(\sigma^2 - \sigma^{-2}) + \cdots a_{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}} \\ &- \sigma^{-\frac{n-2}{2}}) + (1 - a_1(\sigma - \sigma^{-1}) - a_2(\sigma^2 - \sigma^{-2}) - \cdots a_{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}} \\ &- \sigma^{-\frac{n-2}{2}}))\tau \end{aligned}$$

when n is even.
Then, $f + \tau f \tau^{-1} = 0$, hence $v(\tau, f, 1)^2 = 1$.

Example 2

Let $G = C_7 \cdot C_3 = \langle \sigma,\ \tau \mid \sigma^7 = \tau^3 = 1,\ \tau\sigma\tau^{-1} = \sigma^2 \rangle$ be the metacyclic group of order 21.

Let $f = a_0 + a_1\sigma + a_2\sigma^2 + a_3\sigma^3 + a_4\sigma^4 + a_5\sigma^5 + a_6\sigma^6$ be the element in $\mathbf{Z}G$.

Then, $\tau f \tau^{-1} = a_0 + a_1\sigma^2 + a_2\sigma^4 + a_3\sigma^6 + a_4\sigma + a_5\sigma^3 + a_6\sigma^5$ and $\tau^2 f \tau^{-2} = a_0 + a_1\sigma^4 + a_2\sigma + a_3\sigma^5 + a_4\sigma^2 + a_5\sigma^6 + a_6\sigma^3$.

So,

$$\begin{aligned} &f + \tau f \tau^{-1} + \tau^2 f \tau^{-2} \\ &= 3a_0 + (a_1 + a_2 + a_4)(\sigma + \sigma^2 + \sigma^4) + (a_3 + a_5 + a_6)(\sigma^3 + \sigma^5 \\ &+ \sigma^6). \end{aligned}$$

Therefore , $f + \tau f \tau^{-1} + \tau^2 f \tau^{-2} = 0$ if and only if

$$a_0 = a_1 + a_2 + a_4 = a_3 + a_5 + a_6 = 0.$$

Hence , if we set

$$f_0 = a_1\sigma + a_2\sigma^2 + a_3\sigma^3 + (-a_1 - a_2)\sigma^4 + a_5\sigma^5 + (-a_3 - a_5)\sigma^6$$

and

$$v_0 = v(\tau, f_0, 1) = f_0 + (\tau f_0 \tau^{-1} + 1)\tau + (\tau^2 f_0 \tau^{-2})\tau^2,$$

then $v_0^3 = v(\tau, f_0, 1)^3 = 1$.

In particular, when $a_1 = a_2 = a_3 = a_5 = 1$, then

$$f_0 = \sigma + 2\sigma^2 + \sigma^3 - 2\sigma^4 + \sigma^5 - 2\sigma^6$$

and

$$\begin{aligned} v_0 &= v(\tau, f_0, 1) = \sigma + 2\sigma^2 + \sigma^3 - 2\sigma^4 + \sigma^5 - 2\sigma^6 \\ &+ (\sigma^2 + 2\sigma^4 + \sigma^6 - 2\sigma + \sigma^3 - 2\sigma^5 + 1)\tau \\ &+ (\sigma^4 + 2\sigma + \sigma^5 - 2\sigma^2 + \sigma^6 - 2\sigma^3)\tau^2. \end{aligned}$$

5. Proof of the Main Theorem

The purpose of this section is to prove our main theorem.
Let $D_3 = \langle \sigma,\ \tau \mid \sigma^3 = \tau^2 = 1,\ \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the dihedral group of order 6. For $a (\in \mathbf{Q})$, we write $v(a) = a(\sigma - \sigma^{-1})$ $+ \{1 - a(\sigma - \sigma^{-1})\}\tau$
Then, $v(a)^2 = 1$, by Example 1
Note that, when $a \in \mathbf{Z}$, $v(a) \in U(\mathbf{Z}D_3)$.
We show the following:

Main Theorem

(1) $v(1)$ is conjugate to τ in $U(\mathbf{Q}D_3)$, but is not conjugate to τ in $U(\mathbf{Z}D_3)$.

(2) $v(-1)$ is conjugate to τ in $U(\mathbf{Q}D_3)$, but is not conjugate to τ in $U(\mathbf{Z}D_3)$.

(3) $v(1)$ and $v(-1)$ are conjugate in $U(\mathbf{Z}D_3)$.

To prove the theorem , we need the following result

Lemma 1 .
Let $x = a_0 + a_1\sigma + a_2\sigma^2 + b_0\tau + b_1\sigma\tau + b_2\sigma^2\tau$ be the element in $\mathbf{Q}D_3$.
Write $h_1 = a_0 + a_1\sigma + a_2\sigma^2$, $h_2 = b_0 + b_1\sigma + b_2\sigma^2$, $\omega_1 = \sigma - \sigma^{-1}$, and $\omega_2 = \sigma + \sigma^{-1}$.
Then, the following statements are equivalent :

(1) $x\tau = v(a)x$

(2) $h_2 = a\omega_1 h_1 + (1 - a\omega_1)h_2^{(1)}$,
and
$h_1 = a\omega_1 h_2 + (1 - a\omega_1)h_1^{(1)}$.

(3) $h_1 + h_2 = a\omega_1(h_1 + h_2) + (1 - a\omega_1)(h_1^{(1)} + h_2^{(1)})$, (3-1)
and
$h_2 - h_1 = a\omega_1(h_1 - h_2) + (1 - a\omega_1)(h_2^{(1)} - h_1^{(1)})$ (3-2)

(4) $a_1 - a_2 + b_1 - b_2 = 0$, (4-1)
and
$-2aa_0 + (a-1)a_1 + (a+1)a_2 + 2ab_0 + (1-a)b_1 - (1+a)b_2 = 0$ (4-2)

Proof . (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3)
Since
$x\tau = (h_1 + h_2\tau)\tau = h_2 + h_1\tau$, and
$v(a)x = a\omega_1 h_1 + (1 - a\omega_1)h_2^{(1)} + (a\omega_1 h_2 + (1 - a\omega_1)h_1^{(1)})\tau$,
the equivalence of (1), (2) and (3) is clear.

(3) $\Longleftrightarrow$ (4)
First , we show (3-1)$\Longleftrightarrow$ (4-1)
The equality $h_1 + h_2 = a\omega_1(h_1 + h_2) + (1 - a\omega_1)(h_1^{(1)} + h_2^{(1)})$ holds if and only if $(1 - a\omega_1)(h_1 + h_2 - h_1^{(1)} - h_2^{(1)}) = 0$ holds.
But $h_1 - h_1^{(1)} = (a_1 - a_2)\omega_1$ and $h_2 - h_2^{(1)} = (b_1 - b_2)\omega_1$,
so, $h_1 + h_2 - h_1^{(1)} - h_2^{(1)} = (a_1 - a_2 + b_1 - b_2)\omega_1$.
Hence

$$\begin{aligned}&(1 - a\omega_1)(h_1 + h_2 - h_1^{(1)} - h_2^{(1)})\\ &= (1 - a\omega_1)(a_1 - a_2 + b_1 - b_2)\omega_1\\ &= (a_1 - a_2 + b_1 - b_2)(\omega_1 - a\omega_1^2)\\ &= (a_1 - a_2 + b_1 - b_2)(2a + (1-a)\sigma - (1-a)\sigma^2)\end{aligned}$$

Therefore , the equality
$h_1 + h_2 = a\omega_1(h_1 + h_2) + (1 - a\omega_1)(h_1^{(1)} + h_2^{(1)})$
holds if and only if
$a_1 - a_2 + b_1 - b_2 = 0$ holds.

Next , we show (3-2) $\Longleftrightarrow$ (4-2).
Since

$$\begin{aligned}h_2 - h_1 &= a\omega_1(h_1 - h_2) + (1 - a\omega_1)(h_2^{(1)} - h_1^{(1)})\\ &= a\omega_1(h_1 - h_2 - h_2^{(1)} + h_1^{(1)}) + h_2^{(1)} - h_1^{(1)}\end{aligned}$$

we have ,

$$h_2 - h_1 - h_2^{(1)} + h_1^{(1)} = a\omega_1(h_1 - h_2 - h_2^{(1)} + h_1^{(1)}).$$

But $h_1 - h_1^{(1)} = (a_1 - a_2)\omega_1$ and $h_2 - h_2^{(1)} = (b_1 - b_2)\omega_1$,
so, $h_2 - h_1 - h_2^{(1)} + h_1^{(1)} = (b_1 - b_2 - a_1 + a_2)\omega_1$.
On the other hand, since $h_1 + h_1^{(1)} = 2a_0 + (a_1 + a_2)\omega_2$
and $h_2 + h_2^{(1)} = 2b_0 + (b_1 + b_2)\omega_2$, we have
$h_1 - h_2 - h_2^{(1)} + h_1^{(1)} = 2a_0 - 2b_0 + (a_1 + a_2 - b_1 - b_2)\omega_2$
Therefore , we have
$(b_1 - b_2 - a_1 + a_2)\omega_1 = a\omega_1(2a_0 - 2b_0 + (a_1 + a_2 - b_1 - b_2)\omega_2)$
and
$\omega_1(b_1 - b_2 - a_1 + a_2 - 2aa_0 + 2ab_0 - a(a_1 + a_2 - b_1 - b_2)\omega_2) = 0$.
Since $\omega_1\omega_2 = -\omega_1$, we have
$\omega_1(b_1 - b_2 - a_1 + a_2 - 2aa_0 + 2ab_0 + a(a_1 + a_2 - b_1 - b_2)) = 0$
So,
$b_1 - b_2 - a_1 + a_2 - 2aa_0 + 2ab_0 + a(a_1 + a_2 - b_1 - b_2) = 0.$ □

By the results of section 2 , we have

$$\begin{aligned}\mathbf{Q}D_3 &\cong (\mathbf{Q}(\zeta_3)/\mathbf{Q}(\zeta_3 + \zeta_3^{-1}), \tau, 1) \oplus \mathbf{Q}\langle\tau\rangle\\ &\cong (\mathbf{Q}(\zeta_3)/\mathbf{Q}(\zeta_3 + \zeta_3^{-1}), \tau, 1) \oplus \mathbf{Q} \oplus \mathbf{Q}\\ &\cong M_2(\mathbf{Q}(\zeta_3 + \zeta_3^{-1})) \oplus \mathbf{Q} \oplus \mathbf{Q}.\end{aligned}$$

Since $\zeta_3 + \zeta_3^{-1} = -1$, we have

$$\begin{aligned}\mathbf{Q}D_3 &\cong (\mathbf{Q}(\zeta_3)/\mathbf{Q}, \tau, 1) \oplus \mathbf{Q}\langle\tau\rangle\\ &\cong (\mathbf{Q}(\zeta_3)/\mathbf{Q}, \tau, 1) \oplus \mathbf{Q} \oplus \mathbf{Q}\\ &\cong M_2(\mathbf{Q}) \oplus \mathbf{Q} \oplus \mathbf{Q}.\end{aligned}$$

Using this isomorphism, we have the matrix representations T_1, T_2, T_3

$$T_1 : \mathbf{Q}D_3 \to (\mathbf{Q}(\zeta_3)/\mathbf{Q}, \tau, 1)) \to M_2(\mathbf{Q}) ,$$
$$T_2 : \mathbf{Q}D_3 \to M_1(\mathbf{Q}) = \mathbf{Q} ,$$
$$T_3 : \mathbf{Q}D_3 \to M_1(\mathbf{Q}) = \mathbf{Q} ,$$

by

$$T_1(1) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad T_1(\sigma) = \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix},\quad T_1(\sigma^2) = \begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}$$

$$T_1(\tau) = \begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix},\quad T_1(\sigma\tau) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad T_1(\sigma^2\tau) = \begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}$$

$$\begin{aligned}&T_1(a_0 + a_1\sigma + a_2\sigma^2 + b_0\tau + b_1\sigma\tau + b_2\sigma^2\tau)\\ &= \begin{pmatrix}a_0 - a_2 + b_0 - b_2 & -a_1 + a_2 - b_0 + b_1\\ a_1 - a_2 + b_1 - b_2 & a_0 - a_1 - b_0 + b_2\end{pmatrix},\end{aligned}$$

$$T_2(1) = T_2(\sigma) = T_2(\sigma^2) = T_2(\tau) = T_2(\sigma\tau) = T_2(\sigma^2\tau) = 1,$$

$$T_3(1) = T_3(\sigma) = T_3(\sigma^2) = 1,\quad T_3(\tau) = T_3(\sigma\tau) = T_3(\sigma^2\tau) = -1.$$

Further, there is a natural monomorphism

$T = T_1 + T_2 + T_3 : \mathbf{Z}D_3 \longrightarrow (\mathbf{Z}[\zeta_3]/\mathbf{Z}, \tau, 1) \oplus \mathbf{Z}\langle\tau\rangle$
$\subseteq (\mathbf{Z}[\zeta_3]/\mathbf{Z}, \tau, 1) \oplus \mathbf{Z} \oplus \mathbf{Z}$
$\subseteq M_2(\mathbf{Z}) \oplus \mathbf{Z} \oplus \mathbf{Z}.$

Proof of the Main Theorem .

(1) Let $x = a_0 + a_1\sigma + a_2\sigma^2 + b_0\tau + b_1\sigma\tau + b_2\sigma^2\tau$ be an element in $\mathbf{Z}D_3$ such that $x\tau = v(1)x$. Then, by Lemma 1 ,

$a_1 - a_2 + b_1 - b_2 = 0$, and $-2a_0 + 2a_2 + 2b_0 - 2b_2 = 0$.

So,

$$T_1(x) = T_1(a_0 + a_1\sigma + a_2\sigma^2 + b_0\tau + b_1\sigma\tau + b_2\sigma^2\tau)$$
$$= \begin{pmatrix} 2(a_0 - a_2) & -a_1 + a_2 - b_0 + b_1 \\ 0 & a_0 - a_1 - b_0 + b_2 \end{pmatrix}.$$

Since $det(T_1(x)) = 2(a_0 - a_2)(a_0 - a_1 - b_0 + b_2) \neq \pm 1$, where $det(T_1(x))$ is the determinant of $T_1(x)$, we have $T_1(x) \notin GL_2(\mathbf{Z})$. So, we must have $x \notin U(\mathbf{Z}D_3)$. This means that τ and $v(1)$ are not conjugate in $U(\mathbf{Z}D_3)$.

Next, we seek the element $y \in U(\mathbf{Q}D_3)$, such that $y\tau y^{-1} = v(1)$. Let $y = c_0 + c_1\sigma + c_2\sigma^2 + d_0\tau + d_1\sigma\tau + d_2\sigma^2\tau$ be the element in $\mathbf{Q}D_3$ such that $y\tau = v(1)y$.
Then , by Lemma 1 , $c_1 - c_2 + d_1 - d_2 = 0$, and $-2c_0 + 2c_2 + 2d_0 - 2d_2 = 0$. Further, we need the following conditions :

$$det(T_1(y)) \neq 0, \quad det(T_2(y)) \neq 0, \quad det(T_3(y)) \neq 0$$

So, we must have the following :

$c_1 - c_2 + d_1 - d_2 = 0$,
$-2c_0 + 2c_2 + 2d_0 - 2d_2 = 0,$
$2(c_0 - c_2)(c_0 - c_1 - d_0 + d_2) \neq 0$
$c_0 + c_1 + c_2 + d_0 + d_1 + d_2 \neq 0$
$c_0 + c_1 + c_2 - d_0 - d_1 - d_2 \neq 0$

If we set $c_0 = 0,\ c_1 = 0,\ c_2 = 1,\ d_0 = -1,\ d_1 = 1,\ d_2 = 0$, then $y_0 = \sigma^2 - \tau + \sigma\tau$, and these conditions are satisfied. Indeed, we have

$$T_1(y_0) = T_1(\sigma^2 - \tau + \sigma\tau) = \begin{pmatrix} -2 & 3 \\ 0 & 1 \end{pmatrix},$$

$det(T_2(y_0)) = 1$, and $det(T_3(y_0)) = 1$. Hence, $y_0 \in U(\mathbf{Q}D_3)$.

Write $y_0^{-1} = p_0 + p_1\sigma + p_2\sigma^2 + q_0\tau + q_1\sigma\tau + q_2\sigma^2\tau$.

Then,

$$T_1(y_0^{-1}) = T_1(y_0)^{-1} = \begin{pmatrix} -2 & 3 \\ 0 & 1 \end{pmatrix}^{-1} = \tfrac{1}{2}\begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix},$$

$T_2(y_0^{-1}) = 1$, and $T_3(y_0^{-1}) = 1$.

So, we must have

$p_0 - p_2 + q_0 - q_2 = -\frac{1}{2}$
$-p_1 + p_2 - q_0 + q_1 = \frac{3}{2}$
$p_1 - p_2 + q_1 - q_2 = 0$
$p_0 - p_1 - q_0 + q_2 = 1$
$p_0 + p_1 + p_2 + q_0 + q_1 + q_2 = 1$
$p_0 + p_1 + p_2 - q_0 - q_1 - q_2 = 1$

The solution of this equation is

$$p_0 = \frac{1}{2},\ p_1 = 0,\ p_2 = \frac{1}{2},\ q_0 = -\frac{1}{2},\ q_1 = \frac{1}{2},\ q_2 = 0.$$

Therefore

$$y_0^{-1} = \frac{1}{2} + \frac{1}{2}\sigma^2 - \frac{1}{2}\tau + \frac{1}{2}\sigma\tau, \text{ and}$$

$$y_0\tau y_0^{-1} = (\sigma^2 - \tau + \sigma\tau)\tau(\frac{1}{2} + \frac{1}{2}\sigma^2 - \frac{1}{2}\tau + \frac{1}{2}\sigma\tau) = v(1).$$

(2) The proof of (2) is similar to that of (1), so we omit it.

(3) follows from the following equality :

$\tau v(1)\tau^{-1} = v(-1).$ □

References

[1] Curtis C., Reiner I., Representation theory of finite groups and associative algebras (Interscience, New York, 1962).

[2] Reiner I., Maximal Orders, Oxford Science Publications.

[3] Sehgal. S. K. Units in Integral Group Rings, Longman, Essex (1993).

[4] Sekiguchi K., On the unit groups of the integral group rings of metacyclic groups. J. of Pure Appl. Algebra 51 (1988) no3, 311-319.

[5] Sekiguchi K., Units in integral group rings. Proceedings of the 12th Symposium on Ring Theory. (1979).